THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

NEW SERIES

Volume VI *No. 4.* *December 1942*

OFFICE OF THE REGISTRAR OF ...
29 MAY 1943
MADRAS

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PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE.

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ON EXPANSION IN EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEMS III

(The Summability Problem)

BY

S. MINAKSHISUNDARAM, *Madras*.

[Received 3 May 1942]

The scope of this paper is to solve a problem raised by me in Paper I, of the same title.*

Let D be a bounded open domain, simply or multiply connected, whose boundary C is composed of a finite number of regular curves in the (x, y) plane, and let $\omega_n(x, y)$ be the complete normal orthogonal eigenfunctions with the eigenvalues† $\mu_n > 0$, $n = 1, 2, \ldots$ of the boundary value problem

$$\Delta\omega + \mu\omega = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \mu\omega = 0 \tag{1}$$

$$\omega(x, y) = 0 \text{ on } C. \tag{2}$$

Let $f(x, y)$ be any measurable function integrable L in D and let us write

$$a_n = \iint_D f\omega_n \, dxdy \tag{3}$$

$$f(x, y) \sim \sum_{n=1}^{\infty} a_n \omega_n(x, y) \tag{4}$$

the series on the right of (4), where a_n's are obtained by (3), being called the *Fourier series* of $f(x, y)$. Then we

* *Proc. Ind. Acad. Sci.* **12** (1940), 462-5.
For Paper II, see *Jour Ind. Math Soc* **5** (1941), 103-8

† μ_n's are assumed to be in increasing order of magnitude. In I and II the eigenvalues were λ_n, here they are μ_n for the sake of convenience.

raised the question: *Can it be said that at a point of continuity of $f(x,y)$, its Fourier series is summable* $(R, \mu, 1)$?

An answer to this question is contained in Theorems A and B for functions $f(x,y) \in L^2$. If we write, for $p > 0$

$$f_p(r; x,y) = f_p(r) = \frac{p}{\pi r^2} \iint\limits_{C_r} \left(1 - \frac{t^2}{r^2}\right)^{p-1} f(\xi, \eta)\, d\xi d\eta, \qquad (5)$$

where $(\xi - x)^2 + (\eta - y)^2 = t^2 \leqslant r^2$ and C_r is a circle in D with centre (x,y) and radius $r > 0$, then we have

THEOREM A. *If* $f_p(r) \to s$ *as* $r \to 0$, *then* $\Sigma a_n \omega_n(x, y)$ *is summable* (R, μ, k) *to the sum* s *for* $k > \max\left(\frac{3}{2}, p + \frac{1}{2}\right)$.

THEOREM B. *If for a point* (x, y) *in* D, $\Sigma a_n \omega_n(x, y)$ *is summable* (R, μ, k), $k \geqslant 0$, *to the sum* s, *then*

$$\lim_{r=0} f_p(r) = s \text{ for } p > k + \tfrac{1}{2}.$$

For the proofs of these theorems I have adopted K. Chandrasekaran's argument*.

Some fundamental properties of eigenfunctions and eigenvalues are stated in § 1 by the use of which are proved Theorems C and D which are extensions of known theorems. §2 contains the lemmas required for the proofs of Theorems A and B which are given in §§ 3 and 4.

§ 1

SOME FUNDAMENTAL PROPERTIES OF EIGENFUNCTIONS AND EIGENVALUES

$$\text{(i)} \qquad \lim_{n=\infty} \frac{\mu_n}{n} = \frac{4\pi}{A}, \dagger \qquad (6)$$

where A is the area of the domain D, so that the series $\sum_{n=1}^{\infty} \mu_n^{-s}$ is convergent for $s > 1$. If we write

$$\lambda = \mu^{\frac{1}{2}}, \ \lambda_n = \mu_n^{\frac{1}{2}} \qquad (7)$$

then the series $\Sigma \lambda_n^{-s}$ is convergent for $s > 2$.

* In his thesis, *On the Theory of Bessel Summability.*

† Courant, R and Hilbert, D. *Methoden der Math. Physik I*, 379.

(ii) If $G(x, y; \xi, \eta)$ is the Green's function of $\Delta u = 0$ vanishing on the boundary, then

$$\omega_n(x, y) = \mu_n \iint_D G\omega_n d\xi d\eta$$

so that

$$|\omega_n| \leqslant \mu_n \left\{ \iint_D G^2 d\xi d\eta \iint_D \omega_n^2 d\xi d\eta \right\}^{\frac{1}{2}}$$
$$= O(\mu_n) \tag{8}$$

as $n \to \infty$ uniformly in D and C.

(iii) We can prove a little more than (8). For, if $\omega(x, y)$ is a solution of (1), then

$$\omega(x, y)\, J_0(\lambda r) = \frac{1}{2\pi}\int_0^{2\pi} \omega(x + r\cos\theta, y + r\sin\theta)\, d\theta,$$

where $r > 0$ is such that C_r lies wholly in D. We write this in the simpler form

$$\omega J_0(\lambda r) = \frac{1}{2\pi}\int_0^{2\pi} \omega\, d\theta, \tag{9}$$

where $J_0(x)$ is the Bessel function of order zero. Multiplying both sides of (9) by $J_0(\lambda r)r$ and integrating with respect to r from 0 to r, we have

$$\omega \int_0^r t\, J_0^2(\lambda t)\, dt = \frac{1}{2\pi}\iint_{C_r} \omega J_0(\lambda t)\, d\xi d\eta. \tag{10}$$

But

$$\int_0^r t\, J_0^2(\lambda t)\, dt = r^2 \int_0^1 t\, J(\lambda r t)\, dt$$
$$= r^2 \left[\frac{1}{\pi r \lambda} + O\left(\frac{1}{\lambda^2}\right)\right] \text{ as } \lambda \to \infty$$
$$= \frac{r}{\pi\lambda} + O\left(\frac{1}{\lambda^2}\right), \tag{11}$$

and

$$\frac{1}{2\pi}\iint_{C_r} \omega\, J_0(\lambda t)\, d\xi\, d\eta \leqslant \frac{1}{2\pi}\left\{\iint_D \omega^2 d\xi d\eta \iint_{C_r} J_0^2(\lambda t)\, d\xi d\eta\right\}^{\frac{1}{2}}$$

$$= \frac{1}{2\pi}\left\{\int_0^r\int_0^{2\pi} tJ_0^2(\lambda t)\, dt\, d\theta\right\}^{\frac{1}{2}}$$

$$= O(\lambda^{-\frac{1}{2}}) \text{ by (11).} \tag{12}$$

Using (11) and (12) in (10), we see that

$$\omega(x, y) = \frac{\displaystyle\iint_{C_r} \omega\, J_0(\lambda t)\, d\xi d\eta}{2\pi \displaystyle\int_0^r tJ_0^2(\lambda t)\, dt}$$

$$= O(\lambda^{\frac{1}{2}}) = O(\mu^{\frac{1}{4}}). \tag{13}$$

In other words at any interior point (x, y) of D

$$\omega_n(x, y) = O(\lambda_n^{\frac{1}{2}}) = O(\mu_n^{\frac{1}{4}}) = O(n^{\frac{1}{4}}). \tag{14}$$

(iv) From (14) we easily see that

$$\sum \frac{\omega_n(x, y)\ \omega_n(\xi, \eta)}{\mu_n^{\frac{3}{2}+\varepsilon}}$$

is absolutely and uniformly convergent for all points (x, y) and (ξ, η) in the interior of D.

1.2 From (iv) we have

THEOREM C. *If the Fourier coefficients of $f(x, y)$ fulfil the condition*

$$\sum_{n=1}^{\infty} \mu_n^{\nu}\, a_n^2 < \infty$$

for $\nu > 3/2$, then the Fouri r series of $f(x, y)$, viz. $\Sigma a_n \omega_n(x, y)$ is absolutely convergent at every point of D, and in fact uniformly in every domain lying wholly in D.

If $\nu = 2$, the Fourier series is known to be absolutely and uniformly convergent throughout the domain D.

PROOF. By Schwarz's inequality

$$\sum_{n=1}^{\infty} |a_n \omega_n(x, y)| \leqslant \left[\sum_1^{\infty} \mu_n^{\nu} a_n^2 \sum \omega_n^2/\mu_n^{\nu}\right]^{\frac{1}{2}} < \infty$$

if $\nu > \frac{3}{2}$, since each of the series on the right is convergent.

1.3. If $1 \leqslant \nu \leqslant \frac{3}{2}$, we can replace absolute convergence by conditional convergence and prove

THEOREM D. *If* $\sum_{n=1}^{\infty} \mu_n a_n^2 < \infty$, *then the Fourier series of* $f(x, y)$ *converges at a point* (x, y) *in* D *if and only if*

$$\lim_{r\to 0} \frac{1}{\pi r^2} \iint_{C_r} f(\xi, \eta)\, d\xi\, d\eta = \lim_{r\to 0} f_1(r)$$

exists. When this limit exists, the sum of the series is equal to this limit.

PROOF. The sufficiency of the condition has been proved in Paper II. The necessity is also implied in that argument; for, using the notation of that paper (with the difference that we write μ_n for λ_n and r for R)

$$\begin{aligned} |S_n - M_r[f]| &\leqslant |S_n - M_r[S_n]| + M_r[S_n - f] \\ &= O(r^2 \sum^{n} \mu_\nu^2\, a_\nu^2) \\ &= O(r^{-2} \sum_{n+1}^{\infty} a_\nu^2) \\ &= o(1), \end{aligned}$$

provided $\mu_n \leqslant 1/r < \mu_{n+1}$. This proves the theorem.

In particular the Fourier series of $f(x, y)$ will converge at every point of continuity of $f(x, y)$, and it will converge uniformly in every domain in which $f(x, y)$ is continuous.

§ 2. SOME LEMMAS

2.1 First, we give some important properties of Bessel functions.* Let $J_\mu(x)$ denote the Bessel function of order μ, that is

$$J_\mu(x) = \frac{x^\mu}{2^\mu \Gamma(\mu+1)} \left[1 - \frac{x^2}{2(2\mu+2)} + \dots\right]$$

* For these and other properties see G. N. Watson, *Bessel Functions.*

and let

$$\alpha_\mu(x) = \frac{2^\mu \Gamma(\mu+1)}{x^\mu} \mathcal{J}_\mu(x), \; \mu > -1. \tag{15}$$

Then we have

LEMMA 1. *As $t \to 0$, $\alpha_\mu(t) \to 1$ and as $t \to \infty$*

$$\alpha_\mu(t) = O(1/t^{\mu+\frac{1}{2}}). \tag{16}$$

LEMMA 2. *For $0 < \mu < \nu + \frac{1}{2}$*

$$\int_0^\infty \frac{\mathcal{J}_\nu(t)}{t^{\nu-\mu+1}}\, dt = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)}. \tag{17}$$

LEMMA 3. *If $0 < \lambda < \mu+\nu+1$ and if $F(\alpha, \beta, \gamma, x)$ denotes the Hypergeometric series, then*

$$\int_0^\infty \frac{\mathcal{J}_\mu(at)\, \mathcal{J}_\nu(bt)}{t^\lambda}\, dt = \frac{b^\nu \Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda a^{\nu-\lambda+1}\, \Gamma(\nu+1)\Gamma\left(\frac{\mu+\lambda-\nu+1}{2}\right)}$$

$$\times F\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1; b^2/a^2\right), \; 0 < b \leqslant a. \tag{18}$$

LEMMA 4. *If $\nu > \mu > -1$*

$$\alpha_\nu(x) = \frac{2\Gamma(\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu-\mu)} \frac{1}{x^{2\mu+2}} \int_0^x \left(1-\frac{y^2}{x^2}\right)^{\nu-\mu-1} y^{2\mu+1}\alpha_\mu(y)\, dy. \tag{19}$$

2.2 We next give some theorems on summability by typical means. If Σc_n is a given infinite series and (λ_n) a positive increasing sequence of numbers diverging to ∞ and if we write

$$C^k(w) = \sum_{\lambda_n \leqslant w} (w-\lambda_n)^k c_n$$

$$= k\int_0^w (w-t)^{k-1}\, C(t)\, dt, \; k>0$$

$$C(t) = C^0(t)$$

then we say that Σc_n is summable (R, λ, k) to the sum s,

$$\lim_{w=\infty} w^{-k}\, C^k(w) = s.$$

In addition to the first consistency theorem that a series summable (R, λ, k) is also summable (R, λ, k') for $k' > k \geqslant 0$ we require the following second consistency theorem which is a special case of a more general theorem proved by G H Hardy.*

LEMMA 5. *If the divergent sequences* (μ_n) *and* (λ_n) *are defined as in* (7), *then if* Σc_n *is summable* (R, μ, k), $k \geqslant 0$ *to the sum* s, *it is also summable* (R, λ, k) *to* s *and conversely.*

LEMMA 6. *If* $V(x)$ *and* $W(x)$ *be two positive monotonic increasing functions of* x, *each tending to* ∞ *as* $x \to \infty$, *and if*

$$C(x) = O[V(x)]$$

$$C^r(x) = o[W(x)],\ r > 0$$

then

$$C^k(x) = o(V^{1-k/r}\, W^{k/r}),\ 0 < k \leqslant r.$$

2.3 Finally we shall prove some lemmas about the Fourier coefficients of a function $f(x, y)$. If $f(x, y) \in L^2$ in D, we have from (16) of Paper II,

$$S_n(x, y) = M_r[S_n] + \iint_{C_r} \psi_r\, \Delta S_n dxdy,$$

where

$$S_n(x, y) = \sum_{r=1}^{n} a_r \omega_r.$$

Now

$$M_r\,[S_n] = O\left\{\frac{1}{r}\left(\sum^{n} a_\nu^2\right)^{\frac{1}{2}}\right\} = O(1/r)$$

$$\left|\iint_{C_r} \psi_r \Delta S_n\, dxdy\right| = O\left\{r^2 \sum^{n} \mu_\nu^2\, a_\nu^2\right\}^{\frac{1}{2}} = O(r\mu_n).$$

Thus

$$S_n = O(r^{-1}) + O(r\mu_n)$$

$$= O(\lambda_n) \text{ on setting } r = \lambda_n^{-1}.$$

Hence, if we write $c_n = a_n \omega_n(x, y)$ and $\lambda_n \leqslant w < \lambda_{n+1}$

$$C(w) = \sum^{n} a_\nu \omega_\nu = O(w). \tag{20}$$

* *Proc. L.M.S.* **15** (1916), 72.

So we have

LEMMA 7. *If $f(x, y) \varepsilon L^2$ in D and $f(x, y) \sim \Sigma a_n \omega_n$ then*

$$C(w) = O(w).$$

LEMMA 8. *If $f(x, y) \varepsilon L^2$ in D, then*

$$a_n = O(\mu_n) \tag{21}$$

and

$$C(w) = O(w^{\frac{9}{2}}). \tag{22}$$

PROOF.

$$|a_n| = \left|\iint_D f\omega_n \, dx \, dy\right|$$

$$\leqslant \max \omega_n \iint_D |f| \, dx \, dy$$

$$= O(\mu_n) \text{ by } (8),$$

and

$$\sum^n a_\nu w_\nu = \sum^n O(\mu_\nu) \, O(\mu_\nu^{\frac{1}{4}})$$

$$= \sum^n O(\nu^{\frac{5}{4}})$$

$$= O(n^{\frac{9}{4}})$$

so that

$$\sum^n a_\nu w_\nu = O(w^{\frac{9}{2}}).$$

LEMMA 9. *If the Fourier series of a function $f(x, y)$ εL^2 is summable (R, λ, k), $k > 0$, at a point (x, y) in D, to the sum s then*

$$C^r(w) = sw^r + O(w^{r+1-r/k}), \quad 0 < r \leqslant k. \tag{23}$$

This is a consequence of Lemmas 6 and 7. Similarly from Lemmas 6 and 8 we have

LEMMA 10. *If the Fourier series of a function $f(x, y)$ εL in D is summable (R, λ, k), $k > 0$ to the sum s, then*

$$C^r(w) = sw^r + O[w^{\frac{9}{2}(1-r/k)+r}], \quad 0 < r \leqslant k. \tag{24}$$

LEMMA 11. *If $f(x, y) \varepsilon L^2$ in D and if $p > \frac{1}{2}$, then*

$$f_p(r; x, y) = f_p(r) = \sum_{n=1}^{\infty} a_n \omega_n(x, y) \, \alpha_p(\lambda_n r),$$

the series on the right converging absolutely for $r > 0$ such that C_r lies wholly in D. If $p > 1$, the series on the right will be absolutely convergent for all $r > 0$.

PROOF. If C_r lies wholly in D we define $g(\xi, \eta)$ as

$$g(\xi, \eta) = \begin{cases} \frac{p}{\pi r^2}\left(1-t^2/r^2\right)^{p-1} \text{ for } (x-\xi)^2+(y-\eta)^2 = t^2 \leqslant r^2 \\ 0 \text{ for } (\xi, \eta) \text{ outside } C_r. \end{cases}$$

This function $g(\xi, \eta) \in L^2$ in D if $p > \frac{1}{2}$, and so we may apply Parseval's formula and obtain

$$f_p(r) = \iint\limits_D f(\xi, \eta)\, g(\xi, \eta)\, d\xi\, d\eta$$

$$= \frac{p}{\pi r^2} \sum_{n=1}^{\infty} a_n \iint\limits_{C_r} \left(1-\frac{t^2}{r^2}\right)^{p-1} \omega_n(\xi, \eta)\, d\xi\, d\eta$$

$$= \frac{p}{\pi r^2} \sum_{n=1}^{\infty} a_n \omega_n(x, y) \int_0^r \left(1-\frac{t^2}{r^2}\right)^{p-1} t\alpha_0(\lambda_n t)\, dt$$

$$= \Sigma a_n \omega_n \alpha_p(\lambda_n r)$$

the series being absolutely convergent. If $p > 1$, then

$$\sum |a_n \omega_n \alpha_p(\lambda_n r)| \leqslant \left\{\sum a_n^2 \sum \omega_n^2 \alpha_p^2(\lambda_n r)\right\}^{\frac{1}{2}}$$

$$= O\left(\sum_1^{\infty} \mu_n^{\frac{1}{2}} (\lambda_n r)^{-2p-1}\right)^{\frac{1}{2}} = O\left(\frac{1}{r^{2p+1}} \sum_1^{\infty} \frac{1}{n^p}\right)^{\frac{1}{2}} \quad (25)$$

$= O(1)$ for $r > 0$. Hence we get the lemma.

§ 3. PROOF OF THEOREM A.

3.1 Take

$$\nu = \begin{cases} p \text{ if } p > 1 \\ 1+\delta \text{ if } p \leqslant 1, \end{cases} \quad (26)$$

where $\delta > 0$ and arbitrary. Clearly $\nu \geqslant p$. Now if

$$f_p(r) \to s, \text{ as } r \to 0 \quad (27)$$

then

$$f_\nu(r) \to s \text{ as } r \to 0, \quad (28)$$

for it is easily observed that if $\nu > p$

$$f_\nu(r) = \frac{2\Gamma(\nu+1)}{\Gamma(p+1)\,\Gamma(\nu-p)} \frac{1}{r^{2p+2}} \int_0^r \left(1-\frac{t^2}{r^2}\right)^{\nu-p-1} t^{2p+1} f_p(t)\, dt.$$

Also from (26), $\nu > 1$ and hence from Lemma 11

$$f_\nu(r) = \sum_{n=1}^{\infty} a_n \omega_n(x) \alpha_\nu(\lambda_n r) \tag{29}$$

for values of r such that C_r lies wholly in D. But the series on the right of (29) is absolutely convergent for all $r > 0$ by the same lemma. Let the sum of the series be denoted by $\theta_\nu(r)$ which is defined for all $r > 0$, such that when C_r is in D,

$$\theta_\nu(r) \equiv f_\nu(r).$$

Hence from (28)

$$\lim_{r \to 0} \theta_\nu(r) = s \tag{30}$$

and from (25)

$$\theta_\nu(r) = O(1/r^{\nu+\frac{1}{2}}) \text{ as } r \to \infty. \tag{31}$$

3.2 Now consider the function

$$F(x) = x^{2\nu+2} \int_0^\infty t^{2\nu+1} \alpha_\mu(xt)\, \theta_\nu(t)\, dt. \tag{32}$$

From (30), (31) and (16), it is clear that (32) is absolutely convergent if $\mu > \nu + 1$.

3.3 We shall determine the limit of $F(x)$ as $x \to \infty$. From Lemma 2 we see that

$$x^{2\nu+2} \int_0^\infty t^{2\nu+1} \alpha_\mu(xt)\, dt = 2^\mu\, \Gamma(\mu+1) \int_0^\infty \frac{J_\mu(z)}{z^{\mu-2\nu-1}}\, dz$$

$$= \frac{2^{2\nu+1}\Gamma(\nu+1)\,\Gamma(\mu+1)}{\Gamma(\mu-\nu)} = \tau \text{ say.} \tag{33}$$

Again from Lemma 1

$$\int_0^\infty z^{2\nu+1} \,|\alpha_\mu(z)|\, dz < \infty \tag{34}$$

if $\mu > 2\nu + 1\frac{1}{2}$. Now choose a positive number P such that

$$\int_0^\infty z^{2\nu+1} \,|\, \alpha_\mu(z) \,|\, dz < P, \; \max_{0 \leqslant t < \infty} |\, \theta_\nu(t) \,| < P \tag{35}$$

which is possible by virtue of (34), (31) and (30).

Also given $\varepsilon > 0$ we choose a number $\eta > 0$ such that

$$|\, \theta_\nu(t) - s \,| < \frac{\varepsilon}{2P} \text{ for } 0 < t \leqslant \eta. \tag{36}$$

Now, from (32), $F(x)-\tau s$

$$= x^{2\nu+2}\int_0^\infty t^{2\nu+1}\alpha_\mu(xt)\left(\theta_\nu(t)-s\right)dt=\int_0^\eta+\int_\eta^\infty=\mathcal{J}_1+\mathcal{J}_2, \text{say}. \quad (37)$$

$$|\mathcal{J}_1|\leqslant\frac{\varepsilon}{2P}x^{2\nu+2}\int_0^\eta t^{2\nu+1}\,|\,\alpha_\mu(xt)\,|\,dt$$

$$=\frac{\varepsilon}{2P}\int_0^{\eta x} z^{2\nu+1}\,|\,\alpha_\mu(z)\,|\,dz<\frac{\varepsilon}{2}\text{ by (35) and (36).} \quad (38)$$

$$|\mathcal{J}_2|=\left|x^{2\nu+2}\int_\eta^\infty t^{2\nu+1}\alpha_\mu(xt)\theta_\nu(t)s\right|\leqslant 2P\int_{\eta x}^\infty z^{2\nu+1}\,|\alpha_\mu(z)|\,dz.$$

By (35), if $\mu>2\nu+1\frac{1}{2}$, we can choose X so large that for $x>X$

$$|\mathcal{J}_2|\leqslant 2P\int_{\eta x}^\infty z^{2\nu+1}\,|\,\alpha_\mu(z)\,|\,dz<\frac{\varepsilon}{2}. \quad (39)$$

Hence for $x>X$, $|\,F(x)-\tau s\,|<\varepsilon$,

that is
$$\lim_{x\to\infty} F(x)=\tau s. \quad (40)$$

3.4 We shall now evaluate $F(x)$. Since the series for $\theta_\nu(r)$ is absolutely convergent for all $r\geqslant 0$,

$$F(x)=\frac{2^{\nu+\mu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}{x^{\mu-2\nu-2}}\sum_{n=1}^\infty\frac{a_n\omega_n}{\lambda_n^\nu}\int_0^\infty\frac{\mathcal{J}_\mu(\lambda_n t)\,\mathcal{J}_\nu(xt)}{t^{\mu-\nu-1}}\,dt$$

$$=\frac{2^{\nu+\mu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}{x^{\mu-2\nu-2}}\,\frac{1}{2^{\mu-\nu-1}\,x^{2\nu+2-\mu}\,\Gamma(\mu-\nu)}$$

$$\times\sum_{\lambda_n\leqslant x}a_n\omega_n\,F\left(\nu+1,\,\nu-\mu+1,\,\nu+1,\,\lambda_n^2/x^2\right)$$

by Lemma 3

$$=\frac{2^{2\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu-\nu)}\sum_{\lambda_n\leqslant x}a_n\omega_n\left(1-\lambda_n^2/x^2\right)^{\mu-\nu-1}$$

$$=\tau\,x^{-2k}\,C^k(x^2), \quad (41)$$

where $k=\mu-\nu-1$ and τ is as in (33). Hence by (40)

$$\lim_{x=\infty} x^{-2k}C^k(x^2)=s,$$

that is, Σc_n is summable (R,μ,k) for $k=\mu-\nu-1>\nu+\frac{1}{2}$.

3.5 If instead of assuming that $f(x, y) \in L^2$ in Theorem A, we simply assume that $f(x, y) \in L$, then from (21) and (22) we can find a number $r > 0$ so that for $\nu > r$, the series $\Sigma a_n \omega_n \alpha_\nu(\lambda_n r)$ is absolutely convergent for all $r > 0$; and then we can apply the method of Theorem A and assert that

If $f(x, y) \in L$ and if $f_p(r) \to s$ as $r \to 0$, then $\Sigma a_n \omega_n$ is summable (R, μ, k) for $k > \max(r+\frac{1}{2}, p+\frac{1}{2})$.

In particular if *$f(x, y)$ is a continuous function defined in D and vanishing on C, then its Fourier series is summable (R, μ, k), $k > \frac{3}{2}$ at every point in D.* That this summablility is uniform can also be seen; but such problems will be discussed in another paper.

§ 4. Proof of Theorem B.

4.1 Let $h = [k]$ and $s = 0$. Now if $\lambda_n \leqslant w < \lambda_{n+1}$,

$$\sum_{r=1}^{n} a_r \omega_r \alpha_p(\lambda_r t) = C(w)\ \alpha_p(\lambda_n t) + \sum^{n-1} C(\lambda_r) \left\{ \alpha_p(\lambda_r t) - \alpha_p(\lambda_{r+1} t) \right\}$$

$$= C(w)\ \alpha_p(\lambda_n t) - \int_0^{\lambda_n} C(x) \frac{d}{dx}\ \alpha_p(xt)\, dx$$

$$= \sum_{r=0}^{h+1} (-1)^r \left\{ C^r(x) \frac{d^r}{dx^r}\, \alpha_p(xt) \right\}_{x=\lambda_n} + (-1)^{h+2} \int_0^{\lambda_n} C^{h+1}(x) \frac{d^{h+2}}{dx^{h+2}}\, \alpha_p(xt)\ dx.$$

$$= \sum_{r=0}^{h+1} (-1)^r \left\{ C^r(x)\ \frac{d^r}{dx^r}\ \alpha_p(xt) \right\}_{x=\lambda_n} + \frac{(-1)^{h+2}\ \Gamma(h+2)}{\Gamma(k+1)\ \Gamma(1+h-k)} \int_0^{\lambda_n} \frac{d^{h+2}}{dx^{h+2}} \alpha_p(xt) \int_0^x (x-u)^{h-k}\ C^k(u)\ du$$

$$= \sum_{r=0}^{h+1} (-1)^r \left\{ C^r(x)\ \frac{d^r}{dx^r}\ \alpha_p(xt) \right\}_{x=\lambda_n} + \frac{(-1)^{h+2}\ \Gamma(h+2)}{\Gamma(k+1)\ \Gamma(1+h-k)} \int_0^{\lambda_n} C^k(u)\ du \int_u^{\lambda_n} (x-u)^{h-k}\ \frac{d^{h+2}}{dx^{h+2}}\ \alpha_p(xt)\ dx$$

$$= k_1 + k_2 \text{ say.} \tag{42}$$

Here we allow $n \to \infty$, and study the behaviour of k_1 and k_2. We consider the two cases, $k \geqslant 1$ and $k < 1$.

(i) $k \geqslant 1$. In this case if $r \leqslant k$, then $(1-r/k)+r \leqslant k$ and hence by Lemma 9

$$C^r(w) = \begin{cases} o(w^k) \text{ if } r \leqslant k \\ o(w^r) \text{ if } r \geqslant k. \end{cases} \tag{43}$$

(ii) $k < 1$. In this case $(1-r/k)+r$ decreases from 1 to k as r increases from 0 to k, and hence by Lemma 9

$$C^r(w) = \begin{cases} o(w) \text{ if } r < k \\ o(w^r) \text{ if } r \geqslant k. \end{cases} \tag{44}$$

Also $\frac{d^r}{dx^r}\alpha_p(x) = O\left(\frac{1}{x^{p+\frac{1}{2}}}\right)$ as $x \to \infty$, $r = 0, 1, 2, \ldots$ (45)

so that from (43), (44) and (45)

$$C^r(x)\frac{d^r}{dx^r}\alpha_p(x) = \begin{cases} O\left(\frac{1}{x^{p-\frac{1}{2}}}\right) \text{ for } r < k < 1 \\ O\left(\frac{1}{x^{p-k+\frac{1}{2}}}\right) \text{ for } r < k,\ k \geqslant 1 \\ O\left(\frac{1}{x^{p-r+\frac{1}{2}}}\right) \text{ for } r > k. \end{cases} \tag{46}$$

Hence as $n \to \infty$, $k_1 = o(1)$ for $p > k+\frac{1}{2}$, $k \geqslant 0$ (47)

$$k_2 = \frac{(-1)^{h+1}\Gamma(h+2)}{\Gamma(k+1)\Gamma(1+h-k)}\int_0^\infty C^k(u)du\int_u^\infty (x-u)^{h-k}\frac{d^{h+2}}{dx^{h+2}}\alpha_p(xt)dx \tag{48}$$

provided we show that

$$\int_0^{\lambda_n} C^k(u)du\int_{\lambda_n}^\infty (x-u)^{h-k}\frac{d^{h+2}}{dx^{h+2}}\alpha_p(xt)dx = o(1). \tag{49}$$

To prove (49) we have for fixed $t > 0$, by the mean value theorem

$$\left|\int_{\lambda_n}^\infty (x-u)^{h-k}\frac{d^{h+2}}{dx^{h+2}}\alpha_p(xt)\,dx\right|$$

$$\leqslant (\lambda_n-u)^{h-k}\max_{X>\lambda_n}\left|\int_{\lambda_n}^X \frac{d^{h+2}}{dx^{h+2}}\alpha_p(xt)\,dx\right|$$

$$= (\lambda_n-u)^{h-k}O(1/\lambda_n^{p+\frac{1}{2}})$$

and hence

$$\left|\int_0^{\lambda_n} C^k(u)\,du \int_{\lambda_n}^{\infty} (x-u)^{h-k} \frac{d^{h+2}}{dx^{h+2}}\,\alpha_p(xt)\,dx\right|$$

$$= O(\lambda_n^{-p-\frac{1}{2}}) \int_0^{\lambda_n} |\, C^k(u)\,|\,(\lambda_n-u)^{h-k}\,du$$

$$= O(\lambda_n^{-p-\frac{1}{2}}) \int_0^{\lambda_n} o(u^k)\;(\lambda_n-u)^{h-k}du$$

$$= O(\lambda_n^{-p+h+\frac{1}{2}}) = o(1).$$

Thus from (42), (47), (48) and (49) we have for $t > 0$

$$\sum_{n=1}^{\infty} a_n\omega_n\alpha_p(\lambda_n t) = \frac{(-1)^{h+2}\,\Gamma(h+2)}{\Gamma(k+1)\,\Gamma(1+h-k)}$$

$$\times \int_0^{\infty} C_k(u)\,du \int_u^{\infty} (x-u)^{h-k} \frac{d^{h+2}}{dx^{h+2}}\alpha_p(xt)\,dx$$

$$= \frac{(-1)^{h+2}\,\Gamma(h+2)}{\Gamma(k+1)\,\Gamma(1+h-k)}\; t^{h+2}\int_0^{\infty} C^k(u)\;u^{1+h-k}du$$

$$\times \int_1^{\infty} (y-1)^{h-k}\alpha_p^{h+2}(yut)\,dy$$

$$= t^{h+2}\int_0^{\infty} C^k(u)\;u^{1+h-k}I(ut)\,du, \tag{50}$$

where we put

$$I(z) = \frac{(-1)^{k+2}\,\Gamma(h+2)}{\Gamma(k+1)\,\Gamma(1+h-k)} \int_1^{\infty} (y-1)^{h-k}\alpha_p^{h+2}(yz)\,dy. \tag{51}$$

If we assume the following asymptotic formula

$$I(z) = \begin{cases} O[z^{-(h+1-k)}], & |z| \leqslant 1 \\ O[z^{-(p+\frac{1}{2}+h+1-k)}], & |z| \geqslant 1 \end{cases} \tag{52}$$

then

$$t^{h+2}\int_0^{\infty} C^k(u)\;u^{1+h-k}I(ut)\,du = t^{h+2}\int_0^{1/t} + t^{h+2}\int_{1/t}^{\infty}$$

$$= \mathcal{J}_1 + \mathcal{J}_2 \text{ say.} \tag{53}$$

$$\mathcal{J}_1 = t^{h+2}\int_0^{1/t} C^k(u)\;u^{1+h-k}\;O\left\{(ut)^{-(h+1-k)}\right\}du \quad \text{by (52)}$$

$$= O\left(t^{k+1}\int_0^{1/t} C^k(u)\,du\right) = o(1) \text{ as } t \to 0. \tag{54}$$

$$J_2 = t^{h+2}\int_{1/t}^{\infty} C^k(u)\, u^{1+h-k}\, O\left\{(ut)^{-(p+h-k+\frac{3}{2})}\right\}du$$

$$= O(t^{-p+k+\frac{1}{2}})\int_{1/t}^{\infty} C^k(u)\, u^{-p-\frac{1}{2}}\, du$$

$$= t^{-p+k+\frac{1}{2}}\int_{1/t}^{\infty} o(u^{k-p-\frac{1}{2}})\, du = o(1), \text{ if } p > k+\tfrac{1}{2}. \quad (55)$$

We now see from (50) that for $p > k+\frac{1}{2}$,

$$\lim_{t\to 0} \Sigma a_{n\omega_n}\alpha_p(\lambda_n t) = 0 = \lim_{t\to 0} f_p(t)$$

which proves that if $\Sigma a_{n\omega_n}$ is summable (R, λ, k), then for $p > k+\frac{1}{2}$, $f_p(r)$ tends to the same limit as $r \to 0$. But by Lemma 5, if $\Sigma a_{n\omega_n}$ is summable (R, μ, k), it is also summable (R, λ, k) and hence our theorem follows.

It remains to prove (52). This is done as usual by expressing the integral for $I(z)$ as the sum of two integrals, one from 1 to $1+1/z$ and the other from $1+1/z$ to ∞, integrating by parts the second and using the known asymptotic formulae for the derivatives of $\alpha_\mu(x)$*.

4.2 If instead of assuming that $f(x, y) \varepsilon L^2$ we assume that $f(x, y) \varepsilon L$, and use Lemma 10 instead of Lemma 9 in the proof of Theorem B, we obtain:

If $\Sigma a_{n\omega_n}$ is summable (R, μ, k), $k \geqslant 0$, then $\lim_{t\to 0} f_p(t)$ *exists and is equal to the sum of the series for* $p > \max(4, k+\frac{1}{2})$.

Note added on 14—2—1943. The results of this paper formed part of a thesis for which the Ramanujan Memorial Prize was awarded. The method adopted in the thesis was however different; in that method, a class of functions called fractional integrals involving Bessel functions played the role of Young's functions in the theory of Fourier series.

* This is worked out in detail in Mr. K. Chandrasekaran's thesis *loc cit*

THE SECOND THEOREM OF CONSISTENCY FOR ABSOLUTELY SUMMABLE SERIES.

BY

K. CHANDRASEKHARAN, *University of Madras*.

[Received 20 October, 1942]

1. The Second Theorem of Consistency for summable series has been proved by Hardy* in the following form: *If*

(i) *the series* Σc_n *is summable* (λ, k) *to the sum* C,

(ii) μ *is a logarithmico- exponential function of* λ *such that*

$$\mu = O(\lambda^{\Delta}),$$

where Δ *is a constant, then, the series* Σc_n *is summable* (μ, k) *to the sum* C.

The object of this paper is to prove the corresponding theorem† for absolutely summable series.

2. Let (λ_n) be an ascending sequence of positive numbers whose limit is infinity, and let

$$C_\lambda(t) = c_1 + c_2 + \ldots + c_n, \text{ if } \lambda_n < t \leqslant \lambda_{n+1}.$$

Let $$C_\lambda^k(\omega) = C_\lambda(\omega), \text{ if } k = 0,$$

and $$= \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k c_n = k \int_0^\omega C_\lambda(\tau)\ (\omega - \tau)^{k-1}\, d\tau, \text{ if } k > 0.$$

Then, the series Σc_n is said to be absolutely summable (λ, k), or summable $|\lambda, k|$, to the sum C, if,

$$\int_h^\infty \left| \frac{d}{d\omega} \frac{C_\lambda^k(\omega)}{\omega^k} \right| d\omega < \infty.$$

* *Proc London Math Soc.* 15 (1916), 72.

† This theorem which, as far as I know, has not been proved so far, is considered in a thesis for the M. Sc. Degree of the University of Madras.

We prove that,

If Σc_n *is summable* $|\lambda, k|$ *to the sum* C, *and* μ *is a logarithmico-exponential function of* λ *such that* $\mu = O(\lambda^{\Delta})$, *then* Σc_n *is summable* $|\mu, k|$ *to the same sum.*

3. The following lemmas due to Hardy* will be required.

LEMMA 1. *Any L-function* $\mu(\lambda)$ *is continuous, of constant sign, and monotonic, from a certain value of* λ *onwards, and the same is true of any of its derivatives.*

LEMMA 2. *If* $\mu \rightarrow \infty$, *and a number* Δ *exists such that* $\mu = O(\lambda^{\Delta})$, *then* $\mu^{(r)} = O(\mu/\lambda^r)$, $\mu^{(r)}$ *denoting the rth derivative of* $\mu(\lambda)$.

LEMMA 3. *If* μ *satisfies the conditions of Lemma 2, and* v *lies between two positive numbers* g *and* G, *then positive numbers* h *and* H *exist such that*

$$h \leqq \frac{\mu(\lambda v)}{\mu(\lambda)} \leqq H.$$

The same result holds for *decreasing* functions which decrease less rapidly than $\lambda^{-\Delta}$ for some value of Δ.

4. We are given that†

$$R_\lambda^k(\eta) = \frac{k}{\eta^k}\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\eta-\sigma)^{k-1}\,d\sigma \tag{4.1}$$

is of bounded variation in (h, ∞), i.e.

$$\int_h^\infty \left|\frac{d}{d\eta} R_\lambda^k(\eta)\right| d\eta < \infty.$$

We wish to prove that,

$$R_\mu^k(\zeta) = \frac{k}{\zeta^k}\int_{\mu_1}^{\zeta} C_\mu(\tau)\ (\zeta-\tau)^{k-1}\,d\tau \tag{4.2}$$

is of bounded variation in (h, ∞), i.e.

$$\int_h^\infty \left|\frac{d}{d\zeta} R_\mu^k(\zeta)\right| d\zeta < \infty$$

* Second theorem of consistency for summable series, *Proc. L.M.S* 15 (1916)

† We can suppose that $C_\lambda(\sigma) = 0$ for $0 \leqslant \sigma \leqslant \lambda_1$.

or, $$\int_h^\infty \left| \frac{d}{d\eta} R_\mu^k(\zeta) \right| d\eta < \infty.$$

In (4.2) put $\tau = \mu(\sigma)$; since $C_\mu\{\mu(\sigma)\} = C_\lambda(\sigma)$, we obtain

$$R_\mu^k(\zeta) = \frac{k}{\zeta^k} \int_{\lambda_1}^\eta C_\lambda(\sigma)\ (\zeta-\mu)^{k-1}\ \mu' d\sigma,\ \mu = \mu(\sigma),\ \zeta = \mu(\eta). \quad (4.3)$$

5. Proof of the Theorem.

5.1. Let $k > 0$ be an integer. Then,

$$C_\lambda(\sigma) = \frac{1}{k!} \left(\frac{d}{d\sigma}\right)^k C_\lambda^k(\sigma).$$

Therefore, integral (4.3) is a constant multiple of

$$\mathcal{J} = \int_{\lambda_1}^\eta \frac{d}{d\sigma}\ (\zeta-\mu)^k \left(\frac{d}{d\sigma}\right)^k C_\lambda^k(\sigma)\ d\sigma.$$

Integrating by parts and observing that $C_\lambda^k(\sigma)$ and its first $k-1$ derivatives vanish for $\sigma = \lambda_1$, we obtain

$$\mathcal{J} = (-1)^{k-1}\ C_\lambda^k(\eta) \left[\left(\frac{d}{d\sigma}\right)^k (\zeta-\mu)^k\right]_{\sigma=\eta}$$
$$+ (-1)^k \int_{\lambda_1}^\eta C_\lambda^k(\sigma) \left(\frac{d}{d\sigma}\right)^{k+1} (\zeta-\mu)^k\ d\sigma$$
$$= \mathcal{J}_1 + \mathcal{J}_2, \text{ say,}$$

where $$\mathcal{J}_1 = -k!\ C_\lambda^k(\eta)\ (\zeta')^k,$$
ζ' being the value of μ' when $\sigma = \eta$, $\mu = \zeta$.

Hence,

$$\left[\frac{d}{d\eta} R_\mu^\mu(\zeta)\right] = k_1 \left\{ \left(\frac{\eta\zeta'}{\zeta}\right)^k \frac{d}{d\eta} \left(\frac{C_\lambda^k(\eta)}{\eta^k}\right) + \frac{C_\lambda^k(\eta)}{\eta^k} \frac{d}{d\eta} \left(\frac{\eta\zeta'}{\zeta}\right)^k + \frac{d}{d\eta} \frac{\mathcal{J}_2}{\zeta^k} \right\},$$

where k_1 is some constant.

Therefore,

$$\left|\frac{d}{d\eta} R_\mu^k(\zeta)\right| = O(1) \left|\frac{d}{d\eta} R_\lambda^k(\eta)\right| + o(1) \left|\frac{d}{d\eta} \left(\frac{\eta\zeta'}{\zeta}\right)^k\right| + \left|\frac{d}{d\eta} \frac{\mathcal{J}_2}{\zeta^k}\right|. \quad (5.11)$$

By hypothesis, $|d/d\eta\ [R_\lambda^k(\eta)]|$ is integrable in (h, ∞). Now, $d/d\eta\ (\eta\zeta'/\zeta)^k$ is an L-function, and so ultimately of

the same sign, and thus equal to its modulus. Since $\eta\zeta'/\zeta = O(1)$ we observe that $d/d\eta\ (\eta\zeta'/\zeta)^k$ is also absolutely integrable in (h, ∞).

J_2/ζ^k is a sum of constant multiples of integrals of the type

$$\frac{1}{\zeta^r}\int_{\lambda_1}^{\eta} C_\lambda^k(\sigma)\ \mu^s(\mu')^{s_1}\ldots d\sigma \tag{5.12}$$

$$= \frac{1}{\zeta^r}\int_{\lambda_1}^{\eta}\left(\frac{C_\lambda^k(\sigma)}{\sigma^k}\right)\sigma^k\ \mu^s(\mu')^{s_1-1}\ (\mu'')^{s_2}\ldots\mu'\ d\sigma, \tag{5.13}$$

where $0 < s+s_1+s_2+\ldots = r \leqslant k,\ s_1+2s_2+3s_3+\ldots = k+1.$

On integrating (5.13) by parts, we find it equal to

$$\frac{1}{\zeta^r}\frac{C_\lambda^k(\eta)}{\eta^k}\left[\int_{\lambda_1}^{\eta}\sigma^k\mu^s(\mu')^{s_1-1}(\mu'')^{s}\ .\ \mu' d\sigma\right]-\frac{1}{\zeta^r}\int_{\lambda_1}^{\eta}\frac{d}{d\sigma}R_\lambda^k(\sigma)\ .\ \phi(\sigma)\ d\sigma,$$

where

$$\phi(\eta) = \int_{\lambda_1}^{\eta}\sigma^k\ \mu^s(\mu')^{s_1-1}\ (\mu'')^{s_2}\ldots\mu'\ d\sigma.$$

So, $d/d\eta\ (J_2/\zeta^k)$ is a sum of constant multiples of expressions of the type,

$$\frac{d}{d\eta}\left[\frac{\phi(\eta)}{\zeta^r}\ R_\lambda^k(\eta)-\frac{1}{\zeta^r}\int_{\lambda_1}^{\eta}\frac{d}{d\sigma}R_\lambda^k(\sigma)\ .\ \phi(\sigma)\ d\sigma\right] \tag{5.14}$$

$$= R_\lambda^k(\eta)\cdot\frac{d}{d\eta}\frac{\phi(\eta)}{\zeta^r}+\frac{r\zeta'}{\zeta^{r+1}}\int_{\lambda_1}^{\eta}\frac{d}{d\sigma}R_\lambda^k(\sigma).\ \phi(\sigma)\ d\sigma. \tag{5.15}$$

$$\text{Now,}\quad \phi(\eta) = O\int_{\lambda_1}^{\eta}\sigma^k\ \mu^s\ \frac{(\mu)^{s_1-1}\ (\mu)^{s_2}\ldots}{\sigma^{s_1-1}\ \sigma^{2s_2}\ \ldots}\ \mu'\ d\sigma$$

$$= O\int_{\lambda_1}^{\eta}\sigma^{k-s_1-2s_2-\ \ +1}\ \mu^{s+s_1+s_2+\ \ -1}\ \mu'\ d\sigma$$

$$= O\int_{\lambda_1}^{\eta}\mu^{r-1}\ \mu'\ d\sigma = O[\mu^r(\eta)].$$

so that
$$\phi(\eta)/\zeta^r = O(1). \tag{5.16}$$

The expression in (5.15) is, in absolute value, equal to

$$O(1)\left|\frac{d}{d\eta}\frac{\phi(\eta)}{\zeta^r}\right|+\left|r\frac{\zeta'}{\zeta^{r+1}}\int_{\lambda_1}^{\eta}\frac{d}{d\sigma}R_\lambda^k(\sigma)\ .\ \phi(\sigma)\ d\sigma\right|. \tag{5.17}$$

Now, $\phi(\eta)/\zeta^r$ is an L-function which is ultimately of the same sign. Besides, it is bounded, by (5.16). Therefore the first term in (5.17) is absolutely integrable in (h, ∞). As regards the second term, we have

$$\int_h^X \left|\frac{\zeta'}{\zeta^{r+1}}\right| d\eta \int_{\lambda_1}^{\eta} \left|\frac{d}{d\sigma} R_\lambda^k(\sigma).\phi(\sigma)\right| d\sigma$$

$$= \int_h^X \left|\frac{d}{d\sigma} R_\lambda^k(\sigma)\ \phi(\sigma)\right| d\sigma \int_{\eta=\sigma}^{\eta=X} \frac{d\zeta}{\zeta^{r+1}}$$

$$\leqq \int_h^X \left|\frac{d}{d\sigma} R_\lambda^k(\sigma)\right| \frac{|\phi(\sigma)|}{\mu(\sigma)^r} \left(1-\frac{\mu(\sigma)^r}{\mu(X)^r}\right) d\sigma. \tag{5.18}$$

But

$$0 \leqq \frac{1-\mu^r(\sigma)}{\mu^r(X)} \leqq 1,\ \phi(\sigma) = O[\mu(\sigma)]^r$$

so that (5.18) is less than

$$\int_h^X \left|\frac{d}{d\sigma} R_\lambda^k(\sigma)\right| O(1)\ d\sigma < \infty, \text{ as } X \to \infty.$$

Hence

$$\int_h^\infty \left|\frac{d}{d\eta} \frac{J_2}{\zeta^k}\right| d\eta < \infty.$$

Thus the theorem is proved when k is an integer greater than 0.

5.2. Let $0 < k < 1$.

Suppose first that the increase of μ is greater than that of $\log \lambda$, so that $\lambda\mu'$ tends steadily to ∞ with λ.

Let A be any positive constant. Thus, when η is large enough, we can determine a unique η_1, such that $\lambda_1 < \eta_1 < \eta$, $\zeta - \zeta_1 = A\eta_1\zeta'_1$, where $\zeta_1 = \mu(\eta_1)$, $\zeta'_1 = \mu'(\eta_1)$. (5.21).

For, as σ increases from λ_1 to η, $\zeta-\mu$ decreases steadily from $\zeta-\mu_1$ (which is large when η is large) to zero, while $\sigma\mu'$ increases steadily with σ

Suppose that $0 < A < 1$, and that η_1 has been chosen so as to satisfy (5.21). Then, we can determine a positive constant h such that

$$h\eta < \eta_1 < \eta. \tag{5.22}$$

For, $(\zeta-\zeta_1)=(\eta-\eta_1)\zeta'_2$, where ζ'_2 is the value of μ' when σ has a certain value η_2 between η_1 and η. Thus,

$$A\eta_1\,\zeta'_1=\zeta-\zeta_1=(\eta-\eta_1)\,\zeta'_2=\frac{\eta-\eta_1}{\eta_2}\,\eta_2\,\zeta'_2>\frac{\eta-\eta_1}{\eta}\,\eta_1\,\zeta'_1,$$

since $\sigma\mu'$ increases with λ. Hence, $\eta_1>(1-A)\eta$ and we may take $h=1-A$.

We have

$$R^k_\lambda(\eta)=\frac{k}{\eta^k}\int_{\lambda_1}^{\eta}(\eta-\sigma)^{k-1}\,C_\lambda(\sigma)d\sigma$$

$$=\frac{k}{\eta^k}\Big[C_\lambda(\eta)\,\frac{(\eta-\sigma)^k}{-k}\Big]_{\lambda_1}^{\eta}+\frac{k}{\eta^k}\int_{\lambda_1}^{\eta}\frac{(\eta-\sigma)^k}{k}\,dC_\lambda(\sigma).$$

$$=\frac{1}{\eta^k}\int_{\lambda_1}^{\eta}(\eta-\sigma)^k\,dC_\lambda(\sigma), \qquad (5.23)$$

$$\frac{d}{d\eta}\,R^k_\lambda(\eta)=\frac{k}{\eta^{k+1}}\int_{\lambda_1}^{\eta}(\eta-\sigma)^{k-1}\sigma\,dC_\lambda(\sigma). \qquad (5.24)$$

Similarly

$$\frac{d}{d\eta}R^k_\mu(\zeta)=\frac{k\zeta'}{\zeta^{k+1}}\int_{\lambda_1}^{\eta}(\zeta-\mu)^{k-1}\,\mu(\sigma)\,dC_\lambda(\sigma) \qquad (5.241)$$

$$=\frac{k\zeta'}{\zeta^{k+1}}\Big[\int_{\lambda_1}^{\eta_1}+\int_{\eta_1}^{\eta}\Big]=\frac{k\zeta'}{\zeta^{k+1}}\,[J_1+J_2],\text{ say.}$$

Consider J_1.

As σ increases from λ_1 to η, $\zeta-\mu(\sigma)$ decreases* from $\zeta-\mu_1$ to $\zeta-\zeta_1$, and since $k<1$, $[\zeta-\mu]^{k-1}$ increases from $[\zeta-\mu_1]^{k-1}$ to $[\zeta-\zeta_1]^{k-1}=(A\eta_1\zeta'_1)^{k-1}$. Hence,

$$\frac{\zeta'}{\zeta^{k+1}}\,J_1=\frac{\zeta'}{\zeta^{k+1}}\,(A\eta_1\zeta'_1)^{k-1}\int_{\eta_2}^{\eta_1}\mu(\sigma)\,dC_\lambda(\sigma)$$

$$=O\Big(\frac{\zeta'}{\zeta^2}\Big|\int_{\eta_2}^{\eta_1}\mu(\sigma)\,dC_\lambda(\sigma)\Big|\Big)\leq 2k\Big|\frac{d}{d\eta}\,R^1_\mu(\zeta)\Big|. \qquad (5.25)$$

By the First Theorem of Consistency† for absolute summability and by § 5.1, we have,

$$\int_h^\infty\Big|\frac{d}{d\eta}\,R^1_\mu(\zeta)\Big|\,d\eta<\infty.$$

* We assume throughout that $\mu(\sigma)$ is increasing

† Hyslop, J M, On the Absolute Summability of series by Rieszian Means, *Proc. Edin. Math Soc.* 5 (1937—38).

Hence it follows from (5.25) that

$$\int_h^\infty \left| \frac{d}{d\eta} \frac{J_1 \zeta'}{\zeta^{k+1}} \right| d\eta < \infty. \tag{5.251}$$

Next let us consider

$$\frac{\zeta'}{\zeta^{k+1}} \int_{\eta_1}^{\eta} (\zeta-\mu)^{k-1} \mu(\sigma) dC_\lambda(\sigma)$$

$$= \frac{\zeta'}{\zeta^{k+1}} \int_{\eta_1}^{\eta} \left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{k-1} \cdot \frac{\mu(\sigma)}{\sigma} \cdot (\eta-\sigma)^{k-1} \cdot \sigma dC_\lambda(\sigma)$$

$$= \frac{\zeta'}{\eta^{k+1}} \left(\frac{\eta}{\zeta}\right)^{k+1} \int_{\eta_1}^{\eta} \left[\frac{\mu(\eta)-\mu(\sigma)}{\eta-\sigma}\right]^{k-1} \cdot \frac{\mu(\sigma)}{\sigma} \cdot (\eta-\sigma)^{k-1} \sigma \,.\, dC_\lambda(\sigma). \tag{5.26}$$

(i) If $\mu'(\sigma)$ is increasing, then

$$\int_{\eta_1}^{\eta} \left[\frac{\mu(\eta)-\mu(\sigma)}{\eta-\sigma}\right]^{k-1} \frac{\mu(\sigma)}{\sigma} (\eta-\sigma)^{k-1} \sigma dC_\lambda(\sigma)$$

$$= \frac{\mu(\eta)}{\eta} \int_{\eta_2}^{\eta} \left[\frac{\mu(\eta)-\mu(\sigma)}{\eta-\sigma}\right]^{k-1} (\eta-\sigma)^{k-1} \sigma dC_\lambda(\sigma), \quad \eta_1 < \eta_2 < \eta$$

$$= \frac{\mu(\eta)}{\eta} \left[\frac{\mu(\eta_1)-\mu(\eta_2)}{\eta-\eta_2}\right] \int_{\eta_2}^{\eta_3} (\eta-\sigma)^{k-1} \sigma \,.\, dC_\lambda(\sigma), \tag{5.27}$$

since $(\zeta-\mu)/(\eta-\sigma)$ and $\mu(\sigma)/\sigma$ are monotonic, and vary in the same sense as μ'.

(ii) If μ' is decreasing, then

$$\int_{\eta_1}^{\eta} \left[\frac{\mu(\eta)-\mu(\sigma)}{\eta-\sigma}\right]^{k-1} \frac{\mu(\sigma)}{\sigma} \cdot (\eta-\sigma)^{k-1} \sigma \,.\, dC_\lambda(\sigma)$$

$$= \frac{\mu(\eta_1)}{\eta_1} \int_{\eta_1}^{\eta_2} \left[\frac{\mu(\eta)-\mu(\sigma)}{\eta-\sigma}\right]^{k-1} (\eta-\sigma)^{k-1} \sigma \,.\, dC_\lambda(\sigma)$$

$$= \frac{\mu(\eta_1)}{\eta_1} \left[\frac{\mu(\eta)-\mu(\eta_2)}{\eta-\eta_2}\right]^{k-1} \int_{\eta_3}^{\eta_2} (\eta-\sigma)^{k-1} \sigma dC_\lambda(\sigma). \tag{5.28}$$

In either case, $(\zeta-\zeta_2)/(\eta-\eta_2)$ lies between ζ'_1 and ζ'. The order of μ lies between $\log \lambda$ and λ^Δ, and that of μ' between $1/\lambda$ and $\lambda^{\Delta-1}$ (Δ can be evidently supposed to be > 1)* and *a fortiori* between $\lambda^{-\Delta}$ and λ^Δ.

Also, $$h\eta < \eta_1 < \eta.$$

* See Lemma 3 for decreasing functions.

It follows from Lemma 3 that ζ'_1/ζ' lies between fixed positive limits.

Hence we have in any case, from (5.27) and (5.28),

$$\left|\frac{\zeta'}{\zeta^{k+1}} J_2\right| = O\left(\frac{\eta\zeta'}{\zeta}\right)^{k+1}\left|\frac{1}{\eta^{k+1}}\int_{\eta_2}^{\eta_3}(\eta-\sigma)^{k-1}\,\sigma dG_\lambda(\sigma)\right|$$

$$= O(1)\,.\,2k\left|\frac{d}{d\eta}\,R_\lambda^k(\eta)\right|$$

which is integrable in (h, ∞) by hypothesis.

Hence,

$$\int_h^\infty\left|\frac{d}{d\eta}\,\frac{J_2\zeta'}{\zeta^{k+1}}\right|d\eta < \infty. \tag{5.29}$$

We have thus proved the theorem when $0 < k < 1$ and the order of μ lies between $\log\lambda$ and λ^Δ. To extend to all μ without restriction, we have only to repeat the argument of Hardy.

5.3. Let $k > 1$, and non-integral. Let g be the integer such that $g < {}^{\iota} < g+1$. From (5 241),

$$\frac{d}{d\eta}\,R_\mu^k(\zeta) = \frac{k\zeta'}{\zeta^{k+1}}\int_{\lambda_1}^{\eta}(\zeta-\mu)^{k-1}\,\sigma\,.\,\frac{\mu(\sigma)}{\sigma}\,.\,dC_\lambda(\sigma). \tag{5.31}$$

Integrating the right side g times, and noting that

$$\left(\int_{\lambda_1}^{\sigma}\right)^g \sigma dC_\lambda(\sigma) = A\sigma^{g+1}\frac{d}{d\sigma}\,R_\lambda^g(\sigma),$$

where A is a constant, we see that $d/d\eta\; R_\mu^k(\zeta)$ is a constant multiple of

$$\frac{\zeta'}{\zeta^{k+1}}\int_{\lambda_1}^{\eta}\sigma^{g+1}\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\left(\frac{d}{d\sigma}\right)^g\left[(\zeta-\mu)^{k-1}\frac{\mu(\sigma)}{\sigma}\right]d\sigma. \tag{5.32}$$

Now,

$$\left(\frac{d}{d\sigma}\right)^g\left[(\zeta-\mu)^{k-1}\frac{\mu(\sigma)}{\sigma}\right] = \sum_{r=0}^{g} P(r)\,.\left(\frac{d}{d\sigma}\right)^r(\zeta-\mu)^{k-1}$$

$$\times\left(\frac{d}{d\sigma}\right)^{g-r}\left(\frac{\mu(\sigma)}{\sigma}\right). \tag{5.33}$$

But

$$\left(\frac{d}{d\sigma}\right)^r \left[(\zeta-\mu)^{k-1}\right] = \sum_{s=k-1}^{s=k-1+r} (\zeta-\mu)^s \, (\mu')^{s_1} \, (\mu'')^{s_2} \ldots, \tag{5.34}$$

where $s+s_1+s_2+\ldots = k-1$, $s_1+2s_2+3s_3+\ldots = r$,

and

$$\left(\frac{d}{d\sigma}\right)^{g-r} \left[\frac{\mu(\sigma)}{\sigma}\right] = \sum Q(l) \, \mu^{(l)}(\sigma) \, \frac{1}{\sigma^{g-r-l+1}}. \tag{5.35}$$

Substituting (5.34) and (5.35) in (5.33), we have,

$$\left(\frac{d}{d\sigma}\right)^g \left[(\zeta-\mu)^{k-1} \frac{\mu(\sigma)}{\sigma}\right] = \sum\sum P(r) \, Q(l) \, (\zeta-\mu)^s \, (\mu')^{s_1} \, (\mu'')^{s_2}$$

$$\times \ldots \mu^{(l)}(\sigma) \, \frac{1}{\sigma^{g-r-l+1}}. \tag{5.36}$$

Hence, (5.32) is a sum of constant multiples of integrals of the type

$$\frac{\zeta'}{\zeta^{k+1}} \int_{\lambda_1}^{\eta} \sigma^{l+r-1} \,.\, \sigma \,.\, \frac{d}{d\sigma} R_\lambda^g(\sigma) \, (\zeta-\mu)^s \, (\mu')^{s_1} . \ \mu^{(l)}(\sigma) \, d(\sigma) \tag{5.37}$$

$$= \frac{\zeta'}{\zeta^{k+1}} \left[\int_{\lambda_1}^{\eta_1} + \int_{\eta_1}^{\eta}\right] = \frac{\zeta'}{\zeta^{k+1}} \, [\mathcal{J}_1 + \mathcal{J}_2], \text{ say.}$$

We can write $\mathcal{J}_1 \zeta'/\zeta^{k+1}$ as,

$$\frac{\zeta'}{\zeta^{k+1}} \int_{\lambda_1}^{\eta_1} \mu(\sigma) \, \frac{d}{d\sigma} \, R_\lambda^g(\sigma) \, (\zeta-\mu)^s \, \frac{\sigma^{l+r} \, (\mu')^{s_1} (\mu'')^{s_2} \ldots \mu^{(l)}(\sigma)}{\mu(\sigma)} \, d\sigma$$

$$= \frac{\zeta'}{\zeta^{k+1}} \int_{\lambda_1}^{\eta_1} \mu(\sigma) . \frac{d}{d\sigma} R_\lambda^g(\sigma) \, (\zeta-\mu)^s \, \phi(\sigma) d\sigma, \tag{5.38}$$

where

$$\phi(\sigma) = \frac{\sigma^{l+r} (\mu')^{s_1} \, (\mu'')^{s_2} \ldots \mu^{(l)}(\sigma)}{\mu(\sigma)}.$$

$\phi(\sigma)$ is an L-function of σ, and so is continuous, monotonic and of constant sign from a certain value of σ onwards, say λ_2. Then we write (5.38) as

$$\frac{\zeta'}{\zeta^{k+1}} \left[\int_{\lambda_1}^{\lambda_2} + \int_{\lambda_2}^{\eta_1}\right] = \mathcal{J}_{1,1} + \mathcal{J}_{1,2}\,.$$

$\mathcal{J}_{1,1}$ is then a constant multiple of $\frac{\zeta'}{\zeta^{k+1}}$, and $\int_h^\infty \left|\frac{\zeta'}{\zeta^{k+1}}\right| d\eta$ is finite, provided $k>0$, so that $\mathcal{J}_{1,1}$ is absolutely integrable in (h, ∞).

Consider $\mathcal{J}_{1,2}$ now. If $\phi(\sigma)$ is monotonic increasing,

$$\mathcal{J}_{1,2} = \frac{\zeta'}{\zeta^{k+1}}\phi(\eta_1)\int_{\eta_2}^{\eta_1}\mu(\sigma)\,.\,\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\,(\zeta-\mu)^s\,d\sigma,\quad (5.39)$$

$$\lambda_2 < \eta_2 < \eta_1.$$

If $\phi(\sigma)$ is decreasing, then, $\mathcal{J}_{1,2}$ is a constant multiple of

$$\frac{\zeta'}{\zeta^{k+1}}\int_{\lambda_2}^{\eta_2}\mu(\sigma)\,\frac{d}{d\sigma}R_\lambda^g(\sigma)\,(\zeta-\mu)^s\,d\sigma,\ \lambda_2<\eta_2<\eta_1.\quad (5.40)$$

If s is negative, $(\zeta-\mu)^s$ is increasing, and so (5.39) becomes

$$\frac{\zeta'}{\zeta^{k+1}}\phi\,(\eta_1)\,(\zeta-\zeta_1)^s\int_{\eta_3}^{\eta_1}\mu(\sigma)\frac{d}{d\sigma}R_\lambda^g(\sigma)\,d\sigma,\ \eta_2<\eta_3<\eta_1\quad (5.41)$$

and (5.40) becomes

$$\frac{\zeta'}{\zeta^{k+1}}(\zeta-\zeta_2)^s\int_{\lambda_3}^{\eta_2}\mu(\sigma)\frac{d}{d\sigma}R_\lambda^g(\sigma)\,d\sigma,\ \lambda_2<\lambda_3<\eta_2.\quad (5.42)$$

If, on the other hand, s is positive, (5.39) becomes

$$\frac{\zeta'}{\zeta^{k+1}}\phi(\eta_1)\,(\zeta-\zeta_2)^s\int_{\eta_2}^{\eta_3}\mu(\sigma)\frac{d}{d\sigma}R_\lambda^g(\sigma)\,d\sigma,\ \eta_2<\eta_3<\eta_1\quad (5.43)$$

and (5.40) becomes

$$\frac{\zeta'}{\zeta^{k+1}}[\zeta-\mu(\lambda_2)]^s\int_{\lambda_2}^{\eta_4}\mu(\sigma)\frac{d}{d\sigma}[R_\lambda^g(\sigma)]\,d\sigma,\ \lambda_2<\eta_4<\eta_2.\quad (5.44)$$

Now

$$\phi(\eta_1) = \frac{\eta_1^{l+r}\,(\zeta_1')^{s_1}\,(\zeta_1'')^{s_2}\ldots\zeta_1^{(l)}}{\zeta_1}$$

and $\qquad (\zeta-\zeta_2)^s = O(\zeta-\zeta_1)^s = O(\eta_1\zeta_1')^s,$

and $\qquad \zeta^{(r)} = O(\zeta/\eta^r).$

Thus (5.43) becomes

$$\frac{\zeta'}{\zeta^{k+1}}\,O\left(\frac{\eta^{l+r}(\zeta)^{s+s_1+s_2+\cdots}}{\eta^l.\zeta.\eta^{s_1+2s_2+3s_3+\cdots}}\right)\int_{\eta_2}^{\eta_3}\mu(\sigma)\,.\,\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\,d\sigma.$$

$$= O\left(\frac{\zeta'}{\zeta^2}\right)\int_{\eta_2}^{\eta_3}\mu(\sigma)\,.\,\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\,d\sigma,\quad (5.45)$$

and similarly (5.41).

And (5.44) becomes

$$\frac{\zeta'}{\zeta^{k+1-s}}\int_{\lambda_2}^{\eta_4}\mu(\sigma)\cdot\frac{d}{d\sigma}R_\lambda^q(\sigma)\,d\sigma.$$

But, $k+1-s\geqslant 2$ and so (5.44)

$$=O\left(\frac{\zeta'}{\zeta^2}\right)\int_{\lambda_2}^{\eta_4}\mu(\sigma)\cdot\frac{d}{d\sigma}R_\lambda^q(\sigma)\cdot d\sigma, \qquad (5.46)$$

and similarly (5.42).

But

$$\left|\frac{\zeta'}{\zeta^2}\int_{\eta_2}^{\eta_3}\mu(\sigma)\,\frac{d}{d\sigma}R_\lambda^q(\sigma)\,d\sigma\right|\leqq 2\left|\frac{d}{d\eta}R_\mu^{g+1}\,(\zeta)\right|$$

and so (5.45), (5.46) are both absolutely integrable in (h, ∞) since $g+1$ is an integer >1, and since the theorem is proved for integral k.

Hence, $J_1\zeta'/\zeta^{k+1}$ is absolutely integrable in (h, ∞).

It now remains for us to consider $J_2\zeta'/\zeta^{k+1}$. From (5.37) we can easily see that $J_2\zeta'/\zeta^{k+1}$ can also be written in one of the two forms, namely,

$$\eta^{l+r-1}\frac{\zeta'}{\zeta^{k+1}}(\zeta_1')^{s_1}\,(\zeta'')^{s_2}\ldots\zeta^{(l)}\int_{\eta_2}^{\eta}\sigma\cdot\frac{d}{d\sigma}R_\lambda^q(\sigma)\cdot(\zeta-\mu)^s\,d\sigma \qquad (5.47)$$

and

$$\eta_1^{l+r-1}\frac{\zeta'}{\zeta^{k+1}}\,(\zeta_1')^{s_1}\,(\zeta_1'')^{s_2}\ldots\zeta_1^{(l)}\int_{\eta_1}^{\eta_2}\sigma\cdot\frac{d}{d\sigma}R_\lambda^q(\sigma)\cdot(\zeta-\mu)^s\,d\sigma. \qquad (5.48)$$

Let $[s]=k'$. If s is *negative*, we know it is >-1 and $[s]=0$. We can then write (5.47) and (5.48), as

$$\eta^{l+r-1}\,\frac{\zeta'}{\zeta^{k+1}}\,(\zeta')^{s_1}\,(\zeta'')^{s_2}\ldots\zeta^{(l)}\int_{\eta_2}^{\eta_1}\sigma\cdot\frac{d}{d\sigma}R_\lambda^q(\sigma)\cdot\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{s-k'}$$
$$\times(\eta-\sigma)^{s-k'}\,(\zeta-\mu)^{k'}\,d\sigma$$

and

$$\eta_1^{l+r-1}\,\frac{\zeta'}{\zeta^{k+1}}(\zeta_1')^{s_1}\,(\zeta_1'')^{s_2}\ldots\zeta_1^{(l)}\int_{\eta_1}^{\eta_2}\sigma\,\frac{d}{d\sigma}\,R_\lambda^g(\sigma)\,\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{s-k'}$$
$$\times(\eta-\sigma)^{s-k'}\,(\zeta-\mu)^{k'}\,d\sigma.$$

Arguing as in § 5.2, we replace each of these integrals by one of the form

$$(\zeta-\zeta_3)^{k'}\left(\frac{\zeta-\zeta_4}{\eta-\eta_4}\right)^{s-k'}\eta_3^{l+r-1}\frac{\zeta'}{\zeta^{k+1}}(\zeta_3')^{s_1}(\zeta_3'')^{s_2}\ldots\zeta_3^{(l)}$$
$$\times\int_{\eta_5}^{\eta_6}\sigma\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\,(\eta-\sigma)^{s-k'}\,d\sigma, \qquad (5{\cdot}49)$$

where η_3, η_4,...are numbers between η_1 and η, and ζ_3, ζ_4, ..., ζ_3', ζ_4',... are the corresponding values of μ and μ'. (5·49) is less than a constant multiple of

$$\left(\frac{\zeta}{\eta}\right)^s\eta^{l+r-1}\frac{\zeta}{\eta^l}\,\frac{\zeta^{s_1+s_2+}\cdot}{\eta^{s_1+2s_2+}}\cdot\frac{\zeta}{\eta}\,\frac{1}{\zeta^{k+1}}\int_{\eta_5}^{\eta_6}\sigma\frac{d}{d\sigma}R_\lambda^g(\sigma)\,(\eta-\sigma)^{s-k'}\,d\sigma$$
$$=\frac{1}{\eta^{s+2-k'}}\int_{\eta_5}^{\eta_6}\sigma\,.\frac{d}{d\sigma}R_\lambda^g(\sigma)\,(\eta-\sigma)^{s-k'}\,d\sigma$$
$$=O\left(\frac{\zeta'}{\zeta^2}\right)\int_{\eta_5}^{\eta_6}\sigma\frac{d}{d\sigma}\,R_\lambda^g(\sigma)\,(\eta-\sigma)^{s-k'}\,d\sigma$$

the absolute value of which is

$$\leqq 2\left|\frac{d}{d\eta}R_\lambda^{g+s-k'+1}\,(\eta)\right|=2\left|\frac{d}{d\eta}\,R_\lambda^k(\eta)\right|^* \qquad (5{\cdot}50)$$

which by hypothesis is integrable in (h, ∞).

If s is positive, (5·47) and (5·48) are respectively

$$\eta^{l+r-1}\frac{\zeta'}{\zeta^{k+1}}(\zeta')^{s_1}(\zeta'')^{s_2}\ldots\zeta^{(l)}\int_{\eta_2}^{\eta}\sigma\,.\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{s-k'}(\eta-\sigma)^{s-k'-1}$$
$$\times(\zeta-\mu)^{k'+1}\,d\sigma$$

and

$$\eta_1^{l+r-1}\frac{\zeta'}{\zeta^{k+1}}(\zeta'_1)^{s_1}(\zeta''_1)^{s_2}\ldots\zeta_1^{(l)}\int_{\eta_2}^{\eta}\sigma\,.\frac{d}{d\sigma}R_\lambda^g(\sigma)\,.\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{s-k'-1}$$
$$\times\,(\eta-\sigma)^{s-k'-1}(\zeta-\mu)^{k'+1}\,d\sigma.$$

By arguments similar to those used in § 5.2, we replace each of these integrals by one of the type,

$$(\zeta-\zeta_3)^{k'+1}\left(\frac{\zeta-\zeta_4}{\eta-\eta_4}\right)\frac{\zeta'}{\zeta^{k+1}}\eta_3^{l+r-1}(\zeta_3')^{s_1}(\zeta_3'')^{s_2}\ldots\zeta_3^{(l)}$$
$$\times\int_{\eta_5}^{\eta_6}\sigma\frac{d}{d\sigma}\,R_\lambda^g(\sigma)\,.\,(\eta-\sigma)^{s-k'}\,d\sigma,$$

* $s-k'$ = non-integral part, when s is negative $= k-g-1$.

and this is less than a constant multiple of

$$\zeta^{k'+1}\left(\frac{\zeta}{\eta}\right)^{s-k'-1}\frac{\zeta}{\eta\zeta^{k+1}}\cdot\eta^{l+r-1}\frac{\zeta^{s_1+s_2+\dots+1}}{\eta^{l+s_1+2s_2+\dots}}$$

$$\times\int_{\eta_5}^{\eta_6}\sigma\left[\frac{d}{d\sigma}R_\lambda^g(\sigma)\right](\eta-\sigma)^{s-k'-1}\,d\sigma$$

$$=\frac{1}{\eta^{s-k'+1}}\int_{\eta_5}^{\eta_6}\sigma\cdot\frac{d}{d\sigma}R_\lambda^g(\sigma)\,(\eta-\sigma)^{s-k'-1}\,d\sigma.$$

When s is positive

$$s-k'=k-g>0,\quad -1<k-(g+1)<0.$$

Hence, the absolute value of

$$\frac{1}{\eta^{s-k'+1}}\int_{\eta_5}^{\eta_6}\sigma\cdot\frac{d}{d\sigma}R_\lambda^g(\sigma)\cdot(\eta-\sigma)^{s-k'-1}\,d\sigma$$

is less than a constant multiple of

$$\left|\frac{d}{d\eta}R_\lambda^{s-k'+g}(\eta)\right|=\left|\frac{d}{d\eta}R_\lambda^k(\eta)\right|,$$

which is integrable in (h, ∞), by hypothesis.

Hence, $J_2\zeta'/\zeta^{k+1}$ is absolutely integrable in (h, ∞).

The theorem has been proved for non-integral $k>1$ and μ having its order $>\log\lambda$; to cover all cases of μ, we have only to repeat the argument at the end of § 5.2.

I am indebted to Dr. S. Minakshisundaram who suggested this problem to me, and who kindly helped me in solving it.

OFFICE OF THE REGISTRAR OF BOOKS
29 MAY 1943
MADRAS

INTEGRAL REPRESENTATIONS OF WHITTAKER AND WEBER FUNCTIONS*

BY

S. C. DHAR, *Nagpur*

[Received 7 April, 1942]

1. Erdelyi(1) and Meijer(2) gave certain integral representations of Whittaker functions. Erdelyi obtained them by means of fractional integration by parts. In this paper I obtain certain integral representations of Whittaker and Weber functions by double integration.

2. The well-known integral representation(3) of $W_{k,m}(z)$ is given by

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-t} t^{-\frac{1}{2}-k+m} (1+t/z)^{-\frac{1}{2}+k+m} dt, \quad (1)$$

when $R(\frac{1}{2}-k+m) > 0$ and $|z| \neq 0$. We consider the function $W_{k+k',m+m'}(z)$ where k' and m' are any two quantities.

We obtain

$$W_{k+k',m+m'}(z) = \frac{e^{-\frac{1}{2}z} z^{k+k'}}{\Gamma(\frac{1}{2}-k-k'+m+m')} \int_0^\infty e^{-t} t^{-\frac{1}{2}-k-k'+m+m'} \times (1+t/z)^{-\frac{1}{2}+k+k'+m+m} dt \quad (2)$$

when $R(\frac{1}{2}-k-k'+m+m') > 0$, $|z| \neq 0$ and $|\arg z| < \pi$.

But we know

$$\Gamma(-2m'+2k') \left(\frac{t}{z}\right)^{2m'-2k'} = \int_0^\infty e^{-ts/z} s^{-2m'+2k'-1} ds, \quad (3)$$

$R(-2m+2k') > 0$, $R(t/z) > 0$, i.e. if $R(t) > 0$, $R(z) > 0$.

*Read before the Science Congress at Madras, 1940.

Hence by (3) we write (2) in the form

$$W_{k+k',\,m+m'}(z) = \frac{e^{-\frac{1}{2}z}\, z^{k-k'+2m'}}{\Gamma(\frac{1}{2}-k-k'+m+m')\,\Gamma(-2m'+2k')}$$

$$\times \int_0^\infty e^{-t}\, t^{-\frac{1}{2}-k+k'+m-m'}\,(1+t/z)^{-\frac{1}{2}+k+k'+m+m'}\, dt$$

$$\times \int_0^\infty e^{-st/z} s^{-2m'+2k'-1}\, ds. \tag{2'}$$

On changing the order of integration which will be presently justified, the right side becomes

$$\frac{e^{-\frac{1}{2}z}\, z^{2m'+k-k'}}{\Gamma(\frac{1}{2}-k-k'+m+m')\,\Gamma(-2m'+2k')}\int_0^\infty s^{-2m'+2k'-1}\, ds$$

$$\times \int_0^\infty e^{-(1+s/z)\,t}\, t^{-\frac{1}{2}-k+k'+m-m'}\left(1+\frac{t}{z}\right)^{-\frac{1}{2}+k+k'+m+m'} dt. \tag{2''}$$

Using the transformation $(1+s/z)t=y$, (2″) becomes

$$W_{k+k',\,m+m'}(z)$$

$$= \frac{\Gamma(\frac{1}{2}-k-m'+m+k')\, z^{m'-k'}}{\Gamma(\frac{1}{2}-k-k'+m+m')\,\Gamma(-2m'+2k')}\int_0^\infty e^{\frac{1}{2}s}\, s^{-2m'+2k'-1}$$

$$\times\, (1+s/z)^{-\frac{1}{2}-k'-m}\, W_{k+m',\,m+k'}(z+s)\, ds, \text{ by (1)}, \tag{4}$$

$$R(\tfrac{1}{2}-k-m'+m+k')>0.$$

3. It is easy to see that the second integral in (2′) is uniformly convergent and is therefore an analytic function of t for all values of t and z such that $R(t)>0$ and $R(z)>0$, provided $R(-2m'+2k')>0$. Similarly, the second integral in (2″) is uniformly convergent for all finite values of s, provided $R(\frac{1}{2}-k+k'+m-m')>0$. Further we consider the convergence of the last integral in (4). We know[3] that when y is very large,

$$|W_{k,m}(y)| = O(e^{-\frac{1}{2}y}\, y^k),$$

and hence the integrand when s is large $=O(s^{-\frac{3}{2}-m'-m+k+k'})$, and the integral is convergent when $R(\frac{1}{2}+m+m'-k-k')>0$ and $R(-2m+2k')>0$. This justifies the change in the order of integration in § 2.

3·1. Since k, m; k' and m' are arbitrary constants, we can write (4) in the form

$$W_{\alpha,\beta}(z) = \frac{\Gamma(\frac{1}{2}+\delta-\gamma)\, z^{\gamma-\alpha}}{\Gamma(2\alpha-2\gamma)\,\Gamma(\frac{1}{2}+\beta-\alpha)} \int_0^\infty e^{\frac{1}{2}s} s^{2\alpha-2\gamma-1} (1+s/z)^{-\frac{1}{2}-\delta} \times W_{\gamma,\delta}(z+s)\, ds, \quad (6)$$

where $\alpha+\beta=\gamma+\delta$ and $R(\alpha+\gamma)>0$, $R(\frac{1}{2}-\alpha+\beta)>0$, $R(\frac{1}{2}-\gamma+\delta)>0$ and $|z|\neq 0$.

4. Taking

$$\Gamma(-2m'-2k')\,(1+t/z)^{2m'+2k'} = \int_0^\infty e^{-(1+t/s)s} s^{-2m'-2k'-1}\, ds, \quad (7)$$

we next write (2) in the form

$$W_{k+k',m+m'}(z) = \frac{e^{-\frac{1}{2}z} z^{k+k'}}{\Gamma(-2k'-2m')\,\Gamma(\frac{1}{2}-k-k'+m+m')} \times \int_0^\infty e^{-t} t^{-\frac{1}{2}-k-k'+m+m'} \left(1+\frac{t}{z}\right)^{-\frac{1}{2}+k-k'+m-m'} dt \times \int_0^\infty e^{-(1+t/z)s} s^{-2k'-2m'-1} ds.$$

Changing the order of integration as before, we have

$$W_{k+k',m+m'}(z) = \frac{z^{m'+k'}}{\Gamma(-2m'-2k')} \int_0^\infty e^{-\frac{1}{2}s} s^{-2k'-2m'-1} \left(1+\frac{s}{z}\right)^{-\frac{1}{2}-m+k'} \times W_{k-m',m-k'}(z+s)\, ds, \quad (8)$$

$R(-2m'-2k')>0$, $R(\frac{1}{2}-k-k'+m+m')>0$, $|z|\neq 0$.

This can be written in the form

$$W_{\alpha,\beta}(z) = \frac{z^{\alpha-\gamma}}{\Gamma(2\gamma-2\alpha)} \int_0^\infty e^{-\frac{1}{2}s} s^{2\gamma-2\alpha-1} \left(1+\frac{s}{z}\right)^{-\frac{1}{2}-\delta} \times W_{\gamma,\delta}(z+s)\, ds, \quad (9)$$

when $\alpha-\beta=\gamma-\delta$, $R(\gamma-\alpha)>0$, $R(\frac{1}{2}-\alpha+\beta)>0$ and $|z|\neq 0$.

Since $W_{k,m}(z)$ is an even function of m, we can write it in the form[1]

$$W_{k,\lambda}(z) = \frac{z^{k-\mu}}{\Gamma(2\mu-2k)} \int_0^\infty e^{-\frac{1}{2}s} s^{2\mu-2k-1} \left(1+\frac{s}{z}\right)^{-\frac{1}{2}+\nu} \times W_{\mu,\nu}(z+s)\, ds, \quad (10)$$

$k+\lambda=\mu+\nu$, $R(\mu-k)>0$, $R(\frac{1}{2}-k-\lambda)>0$ and $|z|\neq 0$.

If we start with the contour form of the gamma-function (7) we get in place of (9), the integral

$$W_{\alpha,\beta}(z)=\frac{z^{\alpha-\gamma}}{2i\,\Gamma(2\gamma-2\alpha)\sin 2(\alpha-\gamma)\pi}\int_{\infty}^{(0+)} e^{-\frac{1}{2}s}(-s)^{2(\gamma-\alpha)}\times\left(1+\frac{s}{z}\right)^{-\frac{1}{2}-\delta} W_{\gamma,\delta}(z+s)\,ds. \qquad (11)$$

Similarly we can get a contour form of the integral (6).

5. The above method can be applied in the case of parabolic cylinder functions.

We start with Whittaker's integral[4]

$$D_n(z)=\frac{1}{\Gamma(-n)}e^{-\frac{1}{4}z^2}\int_0^\infty e^{-tz-\frac{1}{2}t^2}t^{-n-1}\,dt,\ R(n)<0. \qquad (12)$$

Hence

$$D_{n+m}(z)=\frac{1}{\Gamma(-n-m)}e^{-\frac{1}{4}z^2}\int_0^\infty e^{-tz-\frac{1}{2}t^2}t^{-n-m-1}dt,\ R(n+m)<0. \qquad (13)$$

But

$$\int_0^\infty e^{-st}s^{m-1}ds=\Gamma(m)\,t^{-m},\ R(m)>0,\ R(t)>0. \qquad (14)$$

Hence from (13) and (14) we get

$$D_{n+m}(z)=\frac{1}{\Gamma(-m-n)\,\Gamma(m)}e^{-\frac{1}{4}z^2}\int_0^\infty e^{-zt-\frac{1}{2}t^2}t^{-n-1}dt\int_0^\infty e^{-st}s^{m-1}\,ds.$$

By changing the order of double integrations which is justified and after simplification we get

$$D_{n+m}(z)=\frac{\Gamma(-n)}{\Gamma(-m-n)\,\Gamma(m)}e^{-\frac{1}{4}z^2}\int_0^\infty e^{\frac{1}{4}(z+s)^2}D_n(z+s)\,s^{m-1}\,ds,$$

$$R(m+n)<0,\ R(n)<0 \text{ and } R(m)>0. \qquad (15)$$

The corresponding contour integral form is

$$D_{n+m}(z)=-\frac{\Gamma(-n)}{\Gamma(-m-n)\,\Gamma(m)}\cdot\frac{e^{-\frac{1}{4}z^2}}{2i\sin m\pi}\int_{\infty}^{(0+)} e^{\frac{1}{4}(z+s)^2}\times D_n(z+s)(-s)^{m-1}\,ds. \qquad (15)$$

We write (12) in the form

$$e^{-\frac{1}{4}z^2}D_n(z)=\frac{1}{\Gamma(-n)}\int_0^\infty e^{-\frac{1}{4}(z+t)^2}t^{-n-1}D_0(z+t)\,dt,\ R(n)<0. \qquad (16)$$

Now, when m is an integer,

$$\frac{d^m}{dz^m}\left\{e^{-\frac{1}{4}z^2} D_n(z)\right\} = (-)^m e^{-\frac{1}{4}z^2} D_{n+m}(z).$$

Hence

$$\frac{d^m}{dz^m}\left\{ e^{-\frac{1}{4}z^2} D_n(z) \right\} = \frac{1}{\Gamma(-n)}\int_0^\infty \left[\frac{d^m}{d(z+t)^m}\left\{ e^{-\frac{1}{4}(z+t)^2} D_0(z+t) \right\}\right] \times t^{-n-1}\, dt$$

or $$e^{-\frac{1}{4}z^2} D_{n+m}(z) = \frac{1}{\Gamma(-n)}\int_0^\infty e^{-\frac{1}{4}(z+t)^2} D_n(z+t)\, t^{-n-1}\, dt, \qquad (17)$$

$R(n) < 0$ and m is a positive integer.

The corresponding contour form is[5]

$$e^{-\frac{1}{4}z^2} D_{n+m}(z) = -\frac{\Gamma(n+1)}{2\pi i}\int_\infty^{(+0)} e^{-\frac{1}{4}(z+t)^2} D_m(z+t)\,(-t)^{-n-1} dt. \qquad (18)$$

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ERRATUM

In p. 184, line 1 for $(-s)^{2(\gamma-\alpha)}$ read $(-s)^{2(\gamma-\alpha)-1}$

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ONE PARAMETER GROUPS OF DEFORMATIONS IN RIEMANNIAN SPACES

BY

MOHAMMAD SHABBAR, *Aligarh.*

[Received in revised form, 8 June 1942]

1. INTRODUCTION. Taking a finite one parameter transformation of a Lie group generated by the infinitesimal transformation $D = u^r \partial/\partial x^r$ in a space either Riemannian or of affine connection, Kosambi* was the first to suggest the idea that if the group G operates in that space, the geometric objects could be generated by suitable operators associated with the given Lie group.†

In the present paper it is shown that there exists a unique operator S defined in Riemannian space by

$$ST = u^r T_{\cdot|r} + u^r_{|} T_r + \ldots - u_{|r} T^r_{\cdot} - \ldots \qquad (1)$$

for generating all the components of the tensors of the space, so that, with the well-known notation of Poincare

$$\bar{T}^{ij}_{hk} = e^{\epsilon S} T^{ij}_{hk} \;. \qquad (2)$$

Kosambi's original idea was to investigate the possibilities that are suggested when the equations of Killing are replaced by the vanishing of the S-variation of other fundamental geometric objects, i.e. the Riemann-

* D. D. Kosambi "Continuous groups and two theorems of Euler" *Math. Student*, 2, (1934), 94-100

I express my thanks to Prof D D. Kosambi for his guidance and suggestions during this investigation

† Questions of convergence are ignored throughout, as we assume the relevant series always convergent for all sufficiently small values of the parameter and some suitably restricted region of the space.

Christoffel curvature tensor, Weyl conformal and projective tensors, etc. A greater generalization is not possible as his unpublished notes contain a fairly simple result that under the transformation *inverse* to (1) we have

$$\bar{g}_{pq}(\bar{x})\,\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} \equiv g_{ij}(x),$$

where

$$\bar{g}_{ij}(x) = g_{ij} + \varepsilon S g_{ij} + \ldots + \frac{\varepsilon^n}{n!} S^n g_{ij} + \ldots$$

Thus the space *as a whole* does not change by the Lie transformations, which can always be visualised as a change of co-ordinates. Using the S-operator, it is shown that *isotropic spaces remain isotropic with the same scalar curvature*(1), as would be expected from the above. This is, presumably, the true significance of the well-known result that an isotropic space admits isometric correspondence with $\frac{1}{2}n(n+1)$ parameters, and also admits $\frac{1}{2}n(n+1)$ independent solutions of the equations of Killing.

The emphasis of our investigations is entirely different from that given in standard text books on the theory of continuous groups of transformations, where we see what the conditions are that a space admits a given group. Here we are concerned with both the original and the deformed spaces, their invariants and the relations that subsist between them. Incidentally we have the result:

If a tensor $T^{\cdot\cdot}$ remains invariant under the infinitesimal transformation of a group, it remains invariant under the finite transformation of the group.

The S-operator allows Lie transformations to be defined also for spaces of affine connection without a metric.

2. *The S-operator.*

The original transformation, represented by the operator $D = u^r \partial/\partial x^r$, applies only to point functions as

such. Its first extension, represented by $M = u^r \partial/\partial x^r + \dot{u}^r \partial/\partial \dot{x}^r$, is applied to the metric $F = g_{ij} \dot{x}^i \dot{x}^j$ and we have

$$\bar{F} = F + \varepsilon MF + \dots + \frac{\varepsilon^n}{n!} M^n F + \dots, \tag{2.1}$$

where

$$MF = u^r F_{,r} + \dot{u}^r F_{\cdot r} \tag{2.2}$$

or, in tensorial form

$$MF = u^r F_{|r} + u^r_{|k} \dot{x}^k F_{\cdot r}. \tag{2.3}$$

Here comma, semicolon and solidus denote differentiation w.r.t. x^i, $\dot{x}^i$ and covariant differentiation respectively.

On account of the point of view adopted, the following lemmas are easily proved:

LEMMA 1. $Sdx^i = 0$.

LEMMA 2. $M(A_i dx^i) = (SA_i).dx^i$, *where* $SA_i = u^r A_{i|r} + u^r_{|i} A_r$.

LEMMA 3. $SB^i = u^r B^i_{|r} - u^i_{|r} B^r$.

For $S(A_i B^i) = (SA_i) \cdot B^i + A_i(SB^i) = u^r \dfrac{\partial}{\partial x^r} (A_i B^i)$.

LEMMA 4. ST *is given by the formula* (1).

This is easily deduced from the identity

$$S(T^{ij}_{hk} \; A^h B^k \dots C_i D_j \;) = u^r \frac{\partial}{\partial x^r} (T^{ij}_{hk} \; A^h B^k \dots C_i D_j \;),$$

$A^h, B^k, \dots, C_i, D_j$ being the components of arbitrary vectors.

In particular, we have

THEOREM 1.

$$Sg_{ij} = u_{i|j} + u_{j|i}. \tag{2.4}$$

THEOREM 2. *With the notation of Poincare*

$$\bar{T}^{ij}_{hk} = e^{\varepsilon S} T^{ij}_{hk} \tag{2.5}$$

For let a point $\bar{A}$ be the transform of the point A. Let ε be the parameter of the transformation of the Lie

group. Then the $\bar{T}^{ij}_{hk}$'s are analytic functions of ε. Further, when ε tends to zero

$$\lim \frac{\bar{T}^{ij}_{hk} - T^{ij}_{hk}}{\varepsilon} = ST^{ij\cdot\cdot}_{hk\cdot\cdot} .$$

The theorem follows from this equality, for it is nothing more than the Taylor's series defining $\bar{T}^{ij}_{hk}$

THEOREM 3. *By direct calculation the new* $\bar{\Gamma}$*'s are obtained in the form*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \varepsilon a^i_{jk} + \frac{\varepsilon^2}{2!} S a^i_{jk} + \ldots + \frac{\varepsilon^n}{n!} S^{n-1} a^i_{jk} + \ldots \qquad (2.6)$$

where

$$a^i_{jk} \equiv u^i_{|j|k} + u^\gamma R^i_{j\gamma k} . \qquad (2.7)$$

Since in (2.7), u^i appears only in contravariant form, we can assume the change in Γ's given by (2.6) as defining a finite deformation of any space of symmetric affine connections. The formula (2.7) appears in connection with Eisenhart's study of collineations in an affinely connected space[(2)], but here (2.6) gives the finite transformation of Γ's as well. The vanishing of a^i_{jk} gives the conditions necessary and sufficient for the space to admit the infinitesimal as well as finite affine collineations.

THEOREM 4. *The curvature tensor of the transformed space is given by*

$$\bar{R}^i_{jkl} = e^{\varepsilon S} R^i_{jkl} .* \qquad (2.8)$$

3. We shall prove the invariance of the scalar curvature k_0 for finite transformation of an isotropic Riemann space.

* In a space of affine connection $L^i_{jk} = T^i_{jk} + \Omega^i_{jk}$ the following formulae would be obtained by a similar reasoning.

$$\bar{L}^i_{jk} = L^i_{jk} + \varepsilon b^i_{jk} + \frac{\varepsilon^2}{2!} S b^i_{jk} + \ldots ,$$

where $$b^i_{jk} = u^i_{|j|k} + u^r L^i_{jrk} + 2(u^r \Omega^i_{jr})_{|k}$$

$$ST = u^r T_{|r} + u^r_{|\cdot} T_r + \ldots - u_{|r} T^r - \ldots$$
$$+ 2u^r T_m \Omega^m_r + \ldots - 2u^r T^m \Omega_{mr} - \ldots$$

A necessary and sufficient condition that a space U_n be of constant curvature k_0 is that the components of the fundamental tensor satisfy the conditions

$$R^i_{jkl} = k_0(\delta^i_k g_{jl} - \delta^i_l g_{jk}). \tag{3.1}$$

Substituting (3.1) in (2.8) we have

$$\bar{R}^i_{jkl} = k_0(\delta^i_k g_{jl} - \delta^i_l g_{jk}) + \varepsilon k_0 S(\delta^i_k g_{jl} - \delta^i_l g_{jk}) + \ldots$$
$$= k_0(\delta^i_k \bar{g}_{jl} - \delta^i_l \bar{g}_{jk})$$

which proves the first theorem stated in § 1.

The conditions for the projective and conformal tranformations are respectively

$$SW^i_{jkl} \equiv u^r W^i_{jkl|r} + u^i_{|j} W^i_{rkl} + u^r_{|k} W^i_{jrl} + u^r_{|l} W^i_{jkr} - u^i_{|r} W^r_{jkl} = 0, \tag{3.2}$$

and

$$SC^i_{jkl} \equiv u^r C^i_{jkl|r} + u^i_{|j} C^i_{rkl} + u^r_{|k} C^i_{jrl} + u^r_{|l} C^i_{jkr} - u^i_{|r} C^r_{jkl} = 0. \tag{3.3}$$

These conditions are equivalent to those given by Eisenhart[1] in a less handy form [equations (68.13), (68.14), (69.3) and (69 4)] ; but he has given one more condition associated with his derivation of (3.2), viz.

$$u_m[R^m_{jl k} - R^n_{k|j} + g^{il}(R^m_{kjl|l} - R^m_{jki|l})] = 0. \tag{3.4}$$

Our more powerful notation shows the above expression (3.4) to be identically equal to $SR_{jk} - SR_{kj}$. It is known that for Riemannian spaces $R_{ij} - R_{ji} = 0$, so that Eisenhart's condition (3.4) is identically satisfied. A direct proof can also be given by using the identities of Bianchi and Ricci

It is clear that if the metric tensor remains invariant under the finite transformation of the group, the curvature tensor also remains invariant. For $SR^i_{jkl} = 0$ follows from $Sg_{ij} = 0$. But if $SR^i_{jkl} = 0$, then Sg_{ij} is necessarily equal to zero only for an Einstein space ($R \neq 0$, $n \geqslant 3$) and hence for isotropic spaces also.

PROOF :

$$SR^i_{jkl} = u^r R^i_{jkl|r} + u^r_{|j} R^i_{rkl} + u^r_{|k} R^i_{jrl} + u^r_{|l} R^i_{jkr} - u^i_{|r} R^r_{jkl} = 0.$$

If we contract for i and l we obtain

$$SR_{jk} \equiv u^r R_{jk|r} + u^r_{|j} R_{rk} + u^r_{|k} R_{jr} = 0.$$

For an Einstein space ($R \neq 0$, $n \geqslant 3$) this reduces (because of $R_{ij} = Rg_{ij}/n$ and $R_{|j} = 0$), to

$$u_{j|k} + u_{k|j} \equiv Sg_{jk} = 0.$$

Hence the proposition.

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ON $g(k)$ IN WARING'S PROBLEM

BY

R. K. RUBUGUNDAY, *Madras*

[Received in revised form, 25 December 1942]

1. Dickson* and Pillai† have solved Waring's problem for $g(k)$ in its original form for $k > 6$, provided the criteria

$$(3/2)^k - [(3/2)^k] \neq 1 - 2^{-k}[(3/2)^k] \qquad (1)$$

and

$$(3/2)^k - [(3/2)^k] \neq 1 - 2^{-k}\{[(3/2)^k] + 2\} \qquad (2)$$

both hold. It is conjectured but not proved by Pillai‡ that the criteria (1) and (2) hold when k is even and that Waring's problem for $g(k)$ is solved for even values of k. We show in this paper that (1) holds for all values of k, while (2) can be reduced to a simpler form (Theorem 7). The main theorems of this paper are the following:—

THEOREM A. *Waring's problem for* $g(k)$ *is solved when* $k > 6$, (*a*) *if* k *is even, or* (*b*) *if* k *is a multiple of* 9, *or* (*c*) *if* $2^k - 1$ *is a prime* $\neq 7$, *or* (*d*) *if* $2^k - 1$ *has not all its prime factors* $\equiv 1$ *or* $7 \bmod 24$, *or* (*e*) *if* k *is any multiple of the values satisfying* (*d*), *or* (*f*) *if* k *is a prime, and* $\equiv 3 \bmod 4$, *such that* $2k+1$ *is also a prime, or* (*g*) *if* k *is a multiple of the values satisfying* (*f*).

In particular, since $2^{71} - 1$ has the factor 228479, which is a prime $\equiv -1 \bmod 24$, cases (*d*) and (*e*) cover the value 71, and its multiples. Case (*f*) includes the values for k, 11, 23, 83, 131, 179, 191, 239, 251, etc., while (*g*) includes their multiples.

* *Amer Jour. Math* 58 (1936), 521-9, 530-5.

† *Jour Indian Math Soc.* (2) 2 (1936), 16-44; *Proc. Indian Acad. Sc* 4 (1936), 261

‡ *Jour. Annamalai University*, (1936).

THEOREM B. *Waring's problem for $g(k)$ is solved for all values of k up to n, with exceptions whose number is less than $A+(\log n)/\log\{1+\log(4/3)/\log 3\}$, where A is a constant. Hence Waring's problem for $g(k)$ is solved for almost all values of k.*

2. THEOREM 1. *The criterion (1) is satisfied if $(2^k-1)\nmid(3^k-1)$.*

PROOF. If possible, let

$$(3/2)^k-[(3/2)^k]=1-2^{-k}[(3/2)^k].$$

Then $$3^k=2^k-[(3/2)^k]+2^k[(3/2)^k],$$

or $3^k-1=(2^k-1)\{1+[(3/2)^k]\}$, which contradicts the hypothesis.

COR. (1) *is true for all even values of k.* For when k is even $3=(2^2-1)\mid(2^k-1)$, so that $(2^k-1)\nmid(3^k-1)$.

THEOREM 2. *If b is a quadratic non-residue with respect to some prime factor q of a^k-1, $k>0$ and odd, $a>1$ or <0, $b>1$ or <0, then $(a^k-1)\nmid(b^k-1)$.*

PROOF. The result is obvious when $q\mid b$, since $q\mid(a^k-1)$, and $q\nmid(b^k-1)$. So let us assume that b is prime to q. Now, by Euler's formula, $(q-1)\mid\phi(a^k-1)$. Let

$$\frac{q-1}{2}=\frac{\phi(a^k-1)}{2^\alpha(2m+1)},\ \alpha\geqslant 0,\ m\geqslant 0.$$

Since, $(b/q)\equiv-1$, by hypothesis, we get*

$$b^{(q-1)/2}\equiv-1 \bmod q.$$

Raising both sides to the power of $2m+1$,

$$b^{\{\phi(a^k-1)\}/2^\alpha}\equiv-1 \bmod q.$$

Hence $b^{\{\phi(a^k-1)\}/2^\alpha}-1\equiv-2 \bmod q$, and is thus prime to q, since q is necessarily an odd prime. Now the set of least positive residues of a^ν mod (a^k-1) has the period k

* Dickson: *Introduction to the Theory of Numbers*, p 32 (3)

in x, while Euler's theorem gives the period $\phi(a^k-1)$. It follows that $k \nmid \phi(a^k-1)$ and $k \mid \{\phi(a^k-1)\}/2^a$, since the latter is an integer. Hence,

$$(b^k-1) \mid \{ b^{\{\phi(a^k-1)\}/2^\alpha}-1 \}.$$

Hence b^k-1 is prime to q and hence the theorem.

THEOREM 3. *If k is odd, and if 2^k-1 has a prime factor $\equiv \pm 5$ mod 12, or a prime factor $\equiv \pm 11$ mod 24, then $(2^k-1) \nmid (3^k-1)$.*

PROOF. By the reciprocity law,

$$\left(\frac{3}{12n+5}\right) = \left(\frac{12n+5}{3}\right) = (2/3) = -1$$

$$\left(\frac{3}{12n-5}\right) = -\left(\frac{12n-5}{3}\right) = -(1/3) = -1.$$

Hence, taking $a=2$, $b=3$ in Theorem 2, the first part follows. Next,

$$\left(\frac{6}{24n+11}\right) = \left(\frac{2}{24n+11}\right)\left(\frac{3}{24n+11}\right)$$

$$= (-1)\left[-\left(\frac{24n+11}{3}\right)\right] = (-1)\left[-(2/3)\right] = -1$$

$$\left(\frac{6}{24n-11}\right) = \left(\frac{2}{24n-11}\right)\left(\frac{3}{24n-11}\right)$$

$$= (-1)\left(\frac{24n-11}{3}\right) = (-1)\left(\tfrac{1}{3}\right) = -1.$$

Hence, by Theorem 2, $(2^k-1) \nmid (6^k-1)$.

Hence $(2^k-1) \nmid \{(6^k-1)-3^k(2^k-1)\}$, i.e. $(2^k-1) \nmid (3^k-1)$.

THEOREM 4. *If k is odd, and if 2^k-1 has a prime factor $\equiv -1$ mod 6, or $\equiv -5$ mod 24, or $\equiv -11$ mod 24, then $(2^k-1) \nmid (3^k+1)$.*

The proof is similar to that of Theorem 3, and follows from Theorem 2 by taking $a=2$, $b=-3$ when the prime factor $\equiv -1$ mod 6, and $a=2$, $b=-6$ in the other two cases.

THEOREM 5. *If k is odd and $\neq 1$, criterion (1) is true.*

PROOF. We have $2^k-1 \equiv 1 \bmod 3 \equiv 7 \bmod 3$. Also $2^k-1 \equiv 7 \bmod 8$. Hence $2^k-1 \equiv 7 \bmod 24$, so that not all of the prime factors of 2^k-1 are $\pm 1 \bmod 24$, for then 2^k-1 would itself be $\equiv \pm 1 \bmod 24$. Hence 2^k-1 which is not divisible by 2 or 3, has a prime factor which is either $\equiv \pm 5 \bmod 12$, or $\equiv \pm 11 \bmod 24$.

The theorem now follows by using Theorems 3 and 1.

Combining Cor. Theorem 1, and Theorem 5, we have

THEOREM 6. *The criterion* (1) *is satisfied for all values of* k.

3. THEOREM 7. *Criterion* (2) *is satisfied for* $k \geqslant 3$, *if and only if* $(2^k-1) \nmid (3^k+1)$.

PROOF. From (2), we have

$$3^k-2^k[(3/2)^k] \neq 2^k-[(3/2)^k]-2$$

i.e.

$$3^k+1 \neq (2^k-1)\{1+[(3/2)^k]\}.$$

Hence when $(3^k+1)/(2^k-1)$ is not an integer, (2) holds. Conversely, if (2) holds,

$$\frac{3^k+1}{2^k-1} \neq 1+[(3/2)^k].$$

Now it is easily seen that

$$\frac{3^k+1}{2^k-1}-(3/2)^k = \frac{(3/4)^k\{1+(2/3)^k\}}{\{1-(1/2)^k\}}$$

is a decreasing function of k, whose value is $5/8$ when $k=3$. Hence when $k \geqslant 3$,

$$(3/2)^k < \frac{3^k+1}{2^k-1} < 1+(3/2)^k,$$

so that if $(3^k+1)/(2^k-1)$ is an integer, it is equal to $1+[(3/2)^k]$. But this is impossible when (2) holds. This completes the proof.

COR. 1. *When k is even,* (2) *holds.*

For since $3 \mid (2^k-1)$, $(2^k-1) \nmid (3^k+1)$.

COR. 2. *When k is a multiple of 9, (2) holds.*

On account of Cor. 1, it is sufficient to consider odd values of k. Now,

$$3^{9m}+1=27^{3m}+1=27\cdot 729^{m'}+1\equiv 27\ (-1)^{m'}+1 \bmod 73,$$
where $2m'=3m-1$.

Hence, $3^{9m}+1\not\equiv 0 \bmod 73$, in fact, $3^{9m}+1$ is either $\equiv 28 \bmod 73$, or $\equiv -26 \bmod 73$.

On the other hand, $73=(1+2^3+2^6)\,|\,(2^9-1)\,|\,(2^{9m}-1)$.

Hence, $(2^{9m}-1)\nmid(3^{9m}+1)$. The result follows by Theorem 7.

COR. 3. *(2) holds when k is odd and >3 such that 2^k-1 is prime.*

PROOF. The congruence equation $x^k-1\equiv 0 \bmod (2^k-1)$ has the k roots, $x\equiv 1, 2, 2^2, \ldots, 2^{k-1}$, which are all distinct. Since also 2^k-1 is a prime, setting $f(x)=x^k-1$, we have*

$$f(x)\equiv(x-1)(x-2)\ \ldots\ (x-2^{k-1}) \bmod (2^k-1).$$

We deduce that $f(-3)\not\equiv 0$, for otherwise $-3-2^r\equiv 0 \bmod (2^k-1)$ for some $r, 0\leqslant r<k$. This would mean that $2^k=2^r+4$. Hence either $r=2$, $k=3$, or otherwise, dividing by $2^{\min(r,2)}$, we get an odd number $=$ an even number. It follows that when k is an odd number >3, $(2^k-1)\nmid(3^k+1)$. Hence (2) holds.

COR. 4. *(2) holds when the conditions of Theorem 4 are satisfied.*

COR. 5. *(2) holds when k is odd and 2^k-1 has no all its prime factors either* $\equiv 1 \bmod 24$, *or* $\equiv 7 \bmod 24$. It is easily seen that under these conditions, the conditions of Theorem 4 are satisfied.

COR. 6. *If k satisfies the conditions of Cor. 5, (2) holds for any multiple of k.*

* Hardy and Wright *Theory of Numbers*, p 84 Theorem 108.

For $(2^k-1) \mid (2^{nk}-1)$ for any integral value of n, and hence if not all the factors of 2^k-1 are $\equiv 1$ or $7 \bmod 24$, the same is true for the factors of $2^{nk}-1$.

COR. 7 *If n is positive, such that $4n+3$ and $8n+7$ are primes,* (2) *holds for values of $k=\lambda(4n+3)$, where λ is an integer $\geqslant 1$.*

PROOF. Now n is of one of the forms $3m$, $3m-1$, or $3m-2$. If $n=3m$, $4n+3$ is not a prime, while if $n=3m-2$, $8n+7$ is not prime. Hence $n=3m-1$, and since

$$(8n+7) \mid (2^{\lambda(4n+3)}-1)^*,$$
$$(24m-1) \mid (2^{\lambda(4n+3)}-1).$$

Thus $24m-1$ which by hypothesis is a prime, is a factor of $2^{\lambda(4n+3)}-1$ which is not $\equiv 1$ or $7 \bmod 24$. The result follows by Cor. 5.

4. The proof of Theorem A is now readily completed. Cases (*a*), (*b*), (*c*) follow from Theorem 7, Cor. 1, 2, 3 respectively, while cases (*d*), (*e*), (*f*), (*g*) follow from Cor. 5, 6, 7 of Theorem 7.

We next prove

THEOREM 8. *Given $\varepsilon > 0$, $r \geqslant 3$, if r and $r+s$ are consecutive integral solutions (if any) of the relation $(2^k-1) \mid (3^k+1)$, we can find an $r_0(\varepsilon)$ such that for all $r > r_0(\varepsilon)$,*

$$s > 1-\varepsilon+\frac{(r+1)\log(4/3)}{\log 3}.$$

PROOF. We have

$$\frac{3^{k+1}+1}{2^{k+1}-1} < 3/2 \cdot \frac{3^k+1}{2^k-1}$$

since this reduces to $3^{k+1}+4.2^{k+1} > 5$, which is true for all k. Let $3^r+1 = M(2^r-1)$ and $3^{r+s}+1 = N(2^{r+s}-1)$, where M and N are integers. By repeated applications of the above inequality, we get $N < (\frac{3}{2})^s M$. The right

* Hardy and Wright: *loc. cit.* p. 78, Theorem 101.

side is a fraction, whose denominator is 2^{s-2} at most, since $4|M$, r being necessarily odd. Hence

$$N \leqslant (3/2)^s M - \frac{1}{2^{s-2}},$$

i.e.
$$\frac{3^{r+s}+1}{2^{r+s}-1} \leqslant (3/2)^s \left(\frac{3^r+1}{2^r-1}\right) - \frac{4}{2^s}.$$

Hence

$$(3^{r+s}+1)(2^{r+s}-2^s) \leqslant (3^{r+s}+3^s)(2^{r+s}-1) - 4(2^{r+s}-1)(2^r-1),$$

i.e. $$2^2(4/3)^r \leqslant (3^s-(3/2)^s) + (3+3^s+4.2^{-s})(2/3)^r - \{(3/2)^s + 4.2^{-s}\}(1/3)^r.$$

This shows that s considered as a function of r has not a finite lower limit of indetermination as $r \to \infty$, for otherwise, when $r \to \infty$, the left side $\to \infty$, while the right side has such a lower limit. Hence when $r \to \infty$, $s \to \infty$ and

$$2^2(4/3)^r \leqslant 3^s\{1+o(1)\}$$

i.e.
$$(4/3)^{r+1} \leqslant 3^{s-1}\{1+o(1)\}$$

or
$$s \geqslant 1 + \frac{(r+1)\log(4/3)}{\log 3} + o(1).$$

This completes the proof.

5. We now proceed to p ove Theorem B. With the hypothesis of Theorem 8, we have $s > Ar$, $r > r_0$, $A = \log \frac{4}{3}/\log 3$. The most unfavourable case arises when $r_0, r_0(A+1), r_0(A+1)^2, \ldots$ are all values of s for which $r+s$ is a solution of $(2^k-1)|(3^k+1)$. This gives at most $O(1) + \dfrac{\log(n/r_0)}{\log(A+1)}$ values of $k \leqslant n$. Waring's problem for $g(k)$ is therefore solved for all values of $k \leqslant n$, excepting for a number $\leqslant O(1) + \dfrac{\log n}{\log\left(1+\frac{\log \frac{4}{3}}{\log 3}\right)}$. Since this number is $o(n)$, it follows that Waring's problem for $g(k)$ is solved for *almost* all values of k.

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Vol. VI

1942

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PRINTED BY R. NARAYANASWAMI IYER
AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE, MADRAS.